On Covers of Point Sets in Finite Geometries

This thesis discusses two substructures in finite geometry.

Firstly we investigate k-arcs in projective planes covering a line. In a finite projective plane, a k-arc $\mathcal{K}$ covers a line l_∞ disjoint from it if every point on l_∞ lies on a secant to $\mathcal{K}$. This concept arises from the problem of trying to ascertain the size of the smallest set of elements for which no linear (n, q, t)-perfect hash family exists. In particular, in $PG(2, q)$, the Desarguesian plane of order q, no linear (q^2, q, k)-perfect hash family exists if there is a k-arc covering a line. We are interested in finding k-arcs covering a given line l_∞ such that k is small with respect to q. We obtain a lower bound on the size of such k-arcs and prove that there are only 4 cases where this bound is met. These cases are characterised by the property that every point on l_∞ is covered by exactly one secant to the k-arc. We then consider the generalisation to n-regular k-arcs, where every point on l_∞ is covered by exactly n secants. We show that n is at most $\frac{k}{2}$ and characterise $\frac{k}{2}$-regular k-arcs as hyperovals in planes of even order. In planes of odd order, however, there are no $\frac{k}{2}$-regular k-arcs, but we show that $(\frac{k}{2} - 1, \frac{k}{2})$-regular k-arcs, where half the points on l_∞ lie on $\frac{k}{2} - 1$ secants and the other half on $\frac{k}{2}$ secants, are precisely the ovals in the plane. In addition, we present examples and constructions of families of small k-arcs covering a line in $PG(2, q)$.

We consider also the generalisation of k-arcs covering a line to (k, n)-arcs covering an arbitrary set of points in the plane and obtain lower bounds on k. Furthermore, we show that the concept of k-arcs covering a line can be extended to that of sets of points covering a hyperplane in higher dimensional projective spaces and show that, in fact, a k-arc covering a line in $PG(2, q)$ also covers a hyperplane in $PG(n, q)$ for all $n > 2$.

Secondly we investigate the properties of a certain type of family of planes in $PG(5, q)$ introduced by Yoshiara. This is a family of $q + 3$ planes $\mathcal{E} = \{\pi_o, \ldots, \pi_{q+2}\}$ such that

(1) the set $\mathcal{O}_i = \{\pi_i \cap \pi_j \mid j \in \{0, \ldots, q + 2\} \setminus \{i\}\}$ is a hyperoval in π_i for all $i = 0, \ldots, q + 2$;

(2) any 3 planes in $\mathcal{E}$ span $PG(5, q)$.

This structure may be used to construct a family of Extended Generalised Quadrangles of order $(q + 1, q - 1)$. We are interested principally in the combinatorial and geometric properties of $\mathcal{E}$. We show that the dual of $\mathcal{E}$ also satisfies conditions (1) and (2), and that this leads to new examples. We also present a coordinatisation of $\mathcal{E}$ and prove a necessary and sufficient condition for a set of o-polynomials to determine $\mathcal{E}$.