Zero/Positive Capacities of Two-Dimensional Runlength-Constrained Arrays

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***Abstract*—A binary sequence satisfies a one-dimensional (d_1, k_1, d_2, k_2) runlength constraint if every run of *zeros* has length at least d_1 and at most k_1 and every run of *ones* has length at least d_2 and at most k_2. A two-dimensional binary array is $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$-constrained if it satisfies the one-dimensional (d_1, k_1, d_2, k_2) runlength constraint horizontally and the one-dimensional (d_3, k_3, d_4, k_4) runlength constraint vertically. For given $d_1, k_1, d_2, k_2, d_3, k_3, d_4, k_4$, the two-dimensional capacity is defined as**

$$C(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4) = \lim_{m,n\to\infty} \frac{\log_2 N(m, n \mid d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)}{mn}$$

where

$$N(m, n \mid d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$$

denotes the number of $m \times n$ binary arrays that are $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$-constrained. Such constrained systems may have applications in digital storage applications.

We consider the question for which values of d_i and k_i is the capacity $C(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$ positive and for which values is the capacity zero. The question is answered for many choices of the d_i and the k_i.

***Index Terms*—Capacity, constraint coding, two dimensional.**

I. Introduction

RUNLENGTH-constrained coding is widely used in digital storage applications, particularly magnetic and optical storage devices [7]. Recent developments in optical storage—especially in the area of holographic memory—increase recording density by exploiting the fact that the recording device is a surface. In this new model, the recording data is regarded as two-dimensional, as opposed to the track-oriented one-dimensional recording paradigm. This new approach, however, necessitates the introduction of new types of constraints which are two-dimensional rather than one-dimensional. While the one-dimensional case has been widely explored, results in the two-dimensional case have been slower to arrive. This is mainly due to the fact that imposing constraints in both dimensions makes the coding problem much more difficult. Nevertheless, in the last decade there has been a considerable progress in the study of two-dimensional constraints.

The most commonly considered constraint is the (d, k) constraint: a binary sequence satisfies a one-dimensional (d, k) runlength constraint if every run of *zeros* has length at least d and at most k (and it is assumed that *ones* occur in runs of length 1 unless $d = 0$ in which case *ones* can occur in runs of arbitrary length). A binary sequence satisfies a one-dimensional (d_1, k_1, d_2, k_2) runlength constraint if every run of *zeros* has length at least d_1 and at most k_1 and every run of *ones* has length at least d_2 and at most k_2. Hence, a (d, k)-constrained sequence is a $(d, k, 1, 1)$-constrained sequence $(d > 0)$. A two-dimensional binary array is said to satisfy a $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$ runlength constraint if it satisfies the one-dimensional (d_1, k_1, d_2, k_2) runlength constraint horizontally (i.e., on every row) and the one-dimensional (d_3, k_3, d_4, k_4) runlength constraint vertically (i.e., on every column). Here, the semicolon separates the horizontal and vertical constraints. For convenience, we will say that a binary array satisfies the (d, k) runlength constraint if each row and each column satisfy the (d, k) runlength constraint and that a binary array satisfies the (d_1, k_1, d_2, k_2) runlength constraint if it is (d_1, k_1, d_2, k_2) runlength constrained both horizontally and vertically. We say that a binary array satisfies the $(d_1, k_1; d_3, k_3)$ runlength constraint if each row satisfies the (d_1, k_1, d_1, k_1) runlength constraint and each column satisfies the (d_3, k_3, d_3, k_3) runlength constraint. Finally, we always allow violation of the smallest runlength constraint at the beginning and the end of sequences and at the edges of arrays.

For given $d_1, k_1, d_2, k_2, d_3, k_3, d_4, k_4$, the two-dimensional capacity of the constraint $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$ is defined to be

$$C(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4) = \lim_{m,n\to\infty} \frac{\log_2 N(m, n \mid d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4}{mn}$$

where

$$N(m, n \mid d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$$

denotes the number of binary arrays of size $m \times n$ satisfying a $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$ runlength constraint. That this limit exists can be shown using methods of [9]. We assume throughout that $1 \le d_i \le k_i$, $1 \le i \le 4$. We sometimes refer to this capacity as being the *capacity of the channel*, referring to the assumption that the capacity is utilized to encode information bits onto codewords represented by arrays. We also refer

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to it as being the capacity of the set of arrays satisying the constraint.

Tight lower and upper bounds on $C(1,\infty)$ for binary arrays were given by Calkin and Wilf [2]. One can easily verify that $C(1,\infty) = C(0,1)$ since the complement of a $(1,\infty)$ runlength-constrained sequence is a $(0,1)$ runlength-constrained sequence. These bounds were improved in [6], [12] and extended to three dimensions by Nagy and Zeger [12]. Weeks and Blahut [24] have considered the capacity of the constraint in which all the *ones* are isolated, i.e., each position with a *one* is surrounded by a ring of eight *zeros*. Ashley and Marcus [1] proved that $C(1,2) = 0$. Their result was extended by Kato and Zeger [9] who determined the positive capacity region of (d,k) runlength-constrained arrays. They proved that $C(d,k) > 0$ if and only if $k > d+1$. Kato and Zeger [10] have extended these results to tackle the zero/positive-capacity problem for $(d_1,k_1,1,1;d_3,k_3,1,1)$ runlength-constrained arrays. But, unlike the (d,k) case, the $(d_1,k_1,1,1;d_3,k_3,1,1)$ case is not completely solved. Ito *et al.* [8] have extended some of the results to higher dimensions. Vardy *et al.* [21] gave a construction for conservative arrays, i.e., arrays in which each row and each column has at least one *zero* and at least one *one*. Talyansky *et al.* [20] gave an efficient construction for t-conservative arrays, i.e., arrays in which each row and each column has at least t transitions of the form $0 \to 1$ or t transitions of the form $1 \to 0$. Talyansky [19] has shown how to derive lower bounds on the capacities of (d,k) runlength-constrained arrays from this construction. Talyansky *et al.* [20] also considered balanced arrays, i.e., arrays in which each row and each column is balanced. These arrays were considered also by Ordentlich and Roth [15]. Finally, Etzion and Wei [4] and later Etzion [3] have considered the full two-dimensional $(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4)$ runlength constraint. In [4], a connection between the capacity and the entropy of Markov random field is examined, and in [3] merging of such arrays is considered. Other work concerning the capacity of constrained arrays can be found in [5], [6], [11],[13], [14], [16]–[18], [22], and [23].

In the present work we also consider general $(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4)$ runlength constraints. We examine the following fundamental question about two-dimensional runlength constraints:

> **For which values of $d_1,k_1,d_2,k_2,d_3,k_3,d_4,k_4$ is $C(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4)$ positive and for which values is it equal to zero?**

Most of our proofs of positive capacity results involve explicit, novel constructions demonstrating that there is sufficient flexibility in the selection of arrays to make the capacities positive. Weak lower bounds on capacity can be derived from these constructions. Our proofs of zero capacity results typically involve combinatorial analysis of patterns allowed by constraints.

Our paper is organized as follows. In Section II, we develop some more notation and give background results that we will need. Then, in Section III, we consider the situation where the constraints on *zeros* and *ones* are equal, but where horizontal and vertical constraints may be different. We are able to give a complete characterization of the zero/positive-capacity region in this case. Section IV gives our major result on constraints with zero capacity. In Section V, we study the case where the constraints on *zeros* and *ones* may be different, but where the horizontal and vertical constraints are the same. Our results here include those of [9] as a special case. We explicitly state the set of cases left unresolved. Then, in Section VI, we give some results and constructions for the most general two-dimensional runlength constraints.

II. Basic Results

In the following, we give three simple lemmas which can be easily verified and which will be used repeatedly in our proofs.

Lemma 1:

$$C(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4) \le C(\hat{d}_1,\hat{k}_1,\hat{d}_2,\hat{k}_2;\hat{d}_3,\hat{k}_3,\hat{d}_4,\hat{k}_4)$$

for $\hat{d}_i \le d_i \le k_i \le \hat{k}_i, 1 \le i \le 4$.

Lemma 2:

$$\begin{aligned} C(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4) &= C(d_2,k_2,d_1,k_1;d_4,k_4,d_3,k_3) \\ &= C(d_3,k_3,d_4,k_4;d_1,k_1,d_2,k_2). \end{aligned}$$

Lemma 3: If $C(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4) > 0$ then

$$C(rd_1,rk_1,rd_2,rk_2;sd_3,sk_3,sd_4,sk_4) > 0$$

for any positive integers r and s.

A very important theorem observed by Kato and Zeger [10] is our main tool in proving that the capacity of constraints are positive.

Theorem 4: If $\mathcal{A}$ and $\mathcal{B}$ are two $s \times t$ $(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4)$-constrained arrays such that the 16 arrays defined by

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \qquad X_i \in \{\mathcal{A},\mathcal{B}\}$$

are also $(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4)$-constrained arrays then $C(d_1,k_1,d_2,k_2;d_3,k_3,d_4,k_4) > 0$.

We will refer to the two arrays $\mathcal{A}$ and $\mathcal{B}$ of Theorem 4 as being *compatible* arrays. This theorem is proved by tiling the plane with compatible arrays. The fact that compatible arrays can be placed next to each other in a variety of ways can be used to show that the capacity of the constraint is positive (in fact, at least $1/st$ when the compatible arrays are of size $s \times t$).

Throughout the paper, J_t denotes a $t \times t$ all-*ones* array (a $t \times t$ block of *ones*) and 0_t denotes a $t \times t$ all-*zeros* array. I_t denotes the $t \times t$ identity matrix.

In some of our proofs, we will be interested in how arrays can be populated with *zeros* and *ones*. We will be interested in finite arrays and arrays which are bi-infinite in both dimensions. To this end, we introduce a coordinate system in which each position in an array is identified by a pair of integers (i,j). The first coordinate i of a position is the *column number* of that position, and increases from left to right. The second coordinate j is the *row number* of the position and increases going up vertically.

III. Equal Constraints on Zeros and Ones

Our results in this section pertain to the case where *zeros* and *ones* are constrained in the same way in each dimension, but where the horizontal and vertical constraints may be different.

Theorem 5: Let $d_1, k_1, d_2, k_2, d_3, k_3, d_4, k_4$ be positive integers satisfying $d_i \le k_i, 1 \le i \le 4$. If $d_1 = k_1$ and $d_2 = k_2$ or if $d_3 = k_3$ and $d_4 = k_4$, then $C(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4) = 0$.

Proof:

- If $d_1 = k_1$ and $d_2 = k_2$ then each row of any $m \times n$ array satisfying the constraint consists of runs of d_1 *zeros* alternating with runs of d_2 *ones*, and so is determined relative to the origin by one of $d_1 + d_2$ possible horizontal shifts. Thus,

$$C(d_1, d_1, d_2, d_2; d_3, k_3, d_4, k_4) = \lim_{m,n\to\infty} \frac{\log_2(d_1 + d_2)^m}{mn} = 0.$$

- If $d_3 = k_3$ and $d_4 = k_4$ then we can use the previous case and Lemma 2 to obtain

$$C(d_1, k_1, d_2, k_2; d_3, d_3, d_4, d_4) = 0. \qquad \square$$

Theorem 6: $C(d_1, k_1; d_3, k_3) > 0$ if and only if $k_1 > d_1$ and $k_3 > d_3$.

Proof: We first consider the case where $k_1 = d_1 + 1$ and $k_3 = d_3 + 1$. We use Theorem 4 with the following two compatible $(2d_3 + 1) \times (2d_1 + 1)$ arrays:

$$\mathcal{A} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} X_1^* & X_2 \\ X_3 & X_4 \end{bmatrix}$$

where X_1 is a $(d_3{+}1)\times(d_1{+}1)$ all-*ones* array, X_2 is a $(d_3{+}1)\times d_1$ all-*zeros* array, X_3 is a $d_3\times(d_1{+}1)$ all-*zeros* array, X_4 is a $d_3\times d_1$ all-*ones* array, and X_1^* is a $(d_3{+}1)\times(d_1{+}1)$ array whose entries are *ones* except for a *zero* in the bottom right corner.

Hence, by Theorem 4 we have $C(d_1, d_1+1; d_3, d_3+1) > 0$. If $d_1 = k_1$ or $d_3 = k_3$ then by Theorem 5 we have $C(d_1, k_1; d_3, k_3) = 0$. The theorem now follows immediately from Lemma 1. $\square$

Theorem 6 gives a complete characterization of the zero and positive capacity regions for two-dimensional constraints of the type $(d_1, k_1; d_3, k_3)$.

IV. A Zero Capacity Theorem for General Constraints

This section is dedicated to proving the following theorem, in which the constraints on *ones* horizontally and vertically are rather strict. We will prove results in the next section showing that our theorem is as strong as possible for constraints of this type.

Theorem 7: Let d_1, d_2, d_3, d_4 be positive integers. Then $C(d_1, d_1 + r_1, d_2, d_2; d_3, d_3 + r_3, d_4, d_4) = 0$ whenever $d_1 \ge d_2, d_3 \ge d_4, 0 \le r_1 \le 2d_2 - 1$, and $0 \le r_3 \le 2d_4 - 1$.

Before starting the proof, we make some more definitions and then sketch the proof strategy.

A position (i, j) is said to be adjacent to its eight neighbors $(i-1, j-1), (i-1, j), (i-1, j+1), (i, j-1), (i, j+1), (i+1, j-1), (i+1, j), (i+1, j+1)$. Two positions $(i_1, j_1), (i_2, j_2)$ in a set of positions $\mathcal{X}$ are said to be connected in $\mathcal{X}$ if there is a path $x_1 = (i_1, j_1), x_2, x_3, \ldots, x_n = (i_2, j_2)$ with $x_i \in \mathcal{X}, 1 \le i \le n$, such that x_i and x_{i+1} are adjacent for each $1 \le i \le n-1$. A set of positions $\mathcal{X}$ is said to be *connected* if every pair of positions in $\mathcal{X}$ are connected. A connected set of positions $\mathcal{X}$ is said to be *constant* in an array A if the entries in A in the positions in $\mathcal{X}$ are all *zero* or are all *one*.

An $s \times t$ *block* is defined to be a connected set consisting of all the positions in a rectangle with s rows and t columns. A *constant block* in an array is a block of positions in which the array is constant. A *block of ones* is a constant block in which every position is filled with a *one*. An *isolated block* in an array is defined to be a constant block which is bordered on all four sides by positions of the opposite parity (but where the parities in positions diagonally opposite the corner positions of the block are not specified).

We say that an $s \times t$ block *covers* its s rows and its t columns. Two $s \times t$ blocks are called *row leveled* if together they cover $2s$ consecutive rows. Two $s \times t$ blocks are called *column leveled* if together they cover $2t$ consecutive columns. Two $s \times t$ blocks are called *row matched* if together they cover s rows. Two $s \times t$ blocks are called *column matched* if together they cover t columns. Two $s \times t$ blocks are said to be *partially row intersecting* if together they cover more than s rows and less than $2s$ rows. Two $s \times t$ blocks are said to be *partially column intersecting* if together they cover more than t columns and less than $2t$ columns.

Let $\mathcal{D}$ be a bi-infinite connected set of positions in which each row intersects the set in a nonempty but finite, connected set. Let x_j be the least integer such that $(x_j, j) \in \mathcal{D}$ and y_j be the greatest integer such that $(y_j, j) \in \mathcal{D}$. Then $\mathcal{D}$ is said to be a *diagonal* if one of the two following conditions applies.

- For every row j, $x_{j+1} \ge x_j$ and $y_{j+1} \ge y_j$. This diagonal will be called a *right diagonal*.
- For every row j, $x_{j+1} \le x_j$ and $y_{j+1} \le y_j$. This diagonal will be called a *left diagonal*.

A diagonal $\mathcal{D}$ is said to have *period* (s, t) if, for every position (i, j), we have $(i, j) \in \mathcal{D}$ if and only if $(i + s, i + t) \in \mathcal{D}$.

If all the positions in a diagonal are filled with the same value in some array, we say that the diagonal is a *constant* diagonal in that array. If all the positions are filled with *ones*, we speak of a diagonal of *ones*. A *basic* diagonal is one in which every row contains the same number of positions, and in which the positions in row $j + 1$ of the diagonal are displaced either one position to the right or left of those in row j. The *width* of a basic diagonal is the number of positions in any row of the diagonal. A *block* diagonal is a diagonal consisting of blocks positioned so as to touch at the corners. A *non-block* diagonal is a diagonal that is not a block diagonal.

Until the end of the proof of Theorem 7, all statements will refer to arrays which satisfy the constraint $(d_1, d_1 + r_1, d_2, d_2; d_3, d_3 + r_3, d_4, d_4)$ for integer parameters satisfying $d_1 \ge d_2, d_3 \ge d_4, 0 \le r_1 \le 2d_2 - 1$, and $0 \le r_3 \le 2d_4 - 1$, unless otherwise stated. We will denote this constraint by Θ.

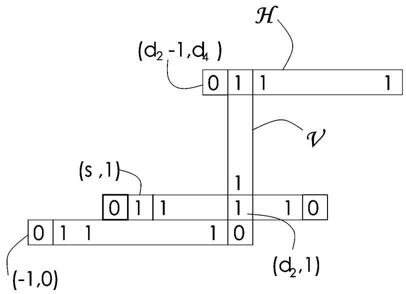


Fig. 1. Illustrating the proof of Lemma 8.

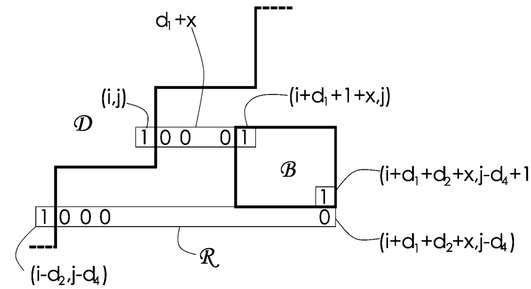


Fig. 2. Illustrating the proof of Lemma 9, Claim 1.

The proof of Theorem 7 will be a result of a sequence of lemmas. First, we will show that every *one* in an array satisfying Θ belongs either to a $d_4 \times d_2$ block or lies on a diagonal which is uniquely determined by d_4 consecutive rows and by d_2 consecutive columns. Because of the constraint on *ones*, any $d_4 \times d_2$ block of *ones* will of course be isolated. We prove that if a diagonal of *ones* exists then every *one* must belong to some diagonal. We show that a channel with the constraint Θ in which every *one* belongs to a diagonal has capacity zero. From then on we consider the case where no diagonals of *ones* exist in the array, so that the array contains only $d_4 \times d_2$ blocks of *ones*. We then show that the capacity of a channel with the constraint Θ in which no pair of $d_4 \times d_2$ blocks of *ones* partially intersects is zero. Finally, we show that if some pair of $d_4 \times d_2$ blocks of *ones* partially intersects, then the capacity must also be zero.

Lemma 8: Every *one* in an array satisfying constraint Θ belongs either to a $d_4 \times d_2$ block of *ones* or to a non-block diagonal of *ones*. Any such diagonal has period (d_2, d_4).

Proof: This proof is illustrated in Fig. 1. Consider a *one* which is not part of a $d_4 \times d_2$ block of *ones* in an array A satisfying constraint Θ. By shifting the array if necessary and applying the horizontal constraint on *ones*, we can assume that there are *ones* in the set of positions $\{(i,0) \,|\, 0 \le i \le d_2 - 1\}$ and that there are *zeros* in positions $(-1,0)$ and $(d_2,0)$. Our special *one* is among these d_2 *ones*.

Suppose that all the positions $\{(i,1) \,|\, 0 \le i \le d_2 - 1\}$ contain *zeros*. Then, from the horizontal constraint, we quickly find that our *one* lies in a $d_4 \times d_2$ block. So some position $(s,1)$ with $0 \le s \le d_2 - 1$ contains a *one*. We assume that this *one* starts a length d_2 run to the right from position $(s,1)$. If this *one* were to start a run to the left, the proof would be very similar, but we would construct a left diagonal. We also have that $s > 0$: if not, we can repeat the argument above. Within d_4 repetitions, we must find an $s > 0$—otherwise, we would construct a $d_4 \times d_2$ block of *ones* containing our special *one*.

So now applying the horizontal constraint on *ones*, we have a run of d_2 *ones* in positions $\{(i,1) \,|\, s \le i \le s + d_2 - 1\}$. This set of positions must include position $(d_2, 1)$. But this position has a *zero* below it, so we must have a vertical run of *ones* $\mathcal{V}$ in positions $\{(d_2, 1+j) \,|\, 0 \le j \le d_4 - 1\}$.

Now consider the position $(d_2 - 1, d_4)$. Suppose this position contains a *one*. Consider column $d_2 - 1$ between rows 0 and d_4. We either have a vertical run of $d_4 + 1$ *ones* in this column (from rows 0 up to d_4 inclusive), contradicting the vertical constraint on *ones*, or this column contains a *zero*, which means it must contain a run of *zeros*. But this run of *zeros* would have length at most $d_4 - 1 \le d_3 - 1$, contradicting the vertical constraint on *zeros*. We conclude that position $(d_2 - 1, d_4)$ contains a *zero*, which means that position (d_2, d_4) initiates a horizontal run $\mathcal{H}$ of d_2 *ones*. This run is located exactly d_2 positions to the right and d_4 positions above our original run of *ones*.

Each of the *ones* in the vertical run $\mathcal{V}$ must also lie in a horizontal run of *ones* of length d_2. These runs cannot start further to the left as we go up $\mathcal{V}$, otherwise, we would obtain a contradiction to the vertical constraint $d_3 \ge d_4$ on *zeros* between that run and the run of *ones* in row 0. These d_4 horizontal runs will form one period of our diagonal.

It is now a simple matter involving repeated application of the horizontal and vertical constraints on *ones* to see that we do indeed have a right diagonal of *ones* of period (d_2, d_4) which contains our original *one*. We omit the details. This diagonal is not a block diagonal because of the fact that the run in row 1 is s positions to the right of the run in row 0, where $1 \le s \le d_2 - 1$. □

Notice that a block diagonal of *ones* also has period (d_2, d_4), so every diagonal of *ones* in an array satisfying constraint Θ has the same period (d_2, d_4). Every such diagonal has exactly d_2 *ones* in each row and d_4 *ones* in each column.

Lemma 9: In any array satisfying constraint Θ, either all the *ones* are located in diagonals or all the *ones* are located in $d_4 \times d_2$ blocks.

Proof: We will prove this lemma via a sequence of claims. Suppose an array satisfies constraint Θ and contains both a diagonal of *ones* $\mathcal{D}$ and a $d_4 \times d_2$ block of *ones* $\mathcal{B}$ which is not part of the diagonal but which is separated from the diagonal only by a run of *zeros*. We prove our claims in the case where $\mathcal{D}$ is a right diagonal and $\mathcal{B}$ is situated to the right of $\mathcal{D}$. The proofs for other configurations are similar.

Claim 1: Every row of the array contains part of a block of *ones* which is separated from the diagonal only by a run of *zeros*. Each block is situated to the left of the block above it.

Proof of Claim 1: We illustrate the proof of Claim 1 in Fig. 2. Assume that the row immediately below block $\mathcal{B}$ does not contain a run of *ones* that is located to the left of $\mathcal{B}$. So this row has a run $\mathcal{R}$ of *zeros* extending horizontally from the diagonal at least as far as the right-hand edge of $\mathcal{B}$. We will calculate the length of $\mathcal{R}$ and show that it violates the constraint Θ.

Let the top row of $\mathcal{B}$ be in row j of the array, and suppose the rightmost *one* of $\mathcal{D}$ in row j is in position (i,j). Then the leftmost *one* of $\mathcal{B}$ in row j must be in position $(i + d_1 + 1 + x, j)$ where $x \ge 0$. Then the bottom right *one* of $\mathcal{B}$ is in position

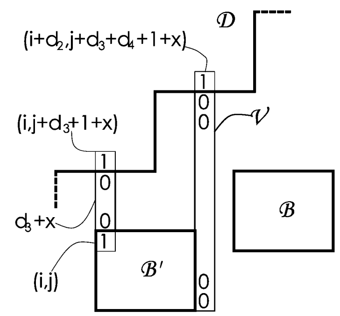


Fig. 3. Illustrating the proof of Lemma 9, Claim 2.

$(i + d_1 + x + d_2, j - d_4 + 1)$. Therefore, the run $\mathcal{R}$ is in row $j - d_4$ and extends from the right-hand edge of $\mathcal{D}$ at least as far as position $(i + d_1 + x + d_2, j - d_4)$.

Now we calculate the position of the rightmost *one* in row $j - d_4$ of $\mathcal{D}$. This diagonal is a right diagonal of period (d_2, d_4) by Lemma 8. Its rightmost *one* in row j is in position (i, j). It follows that its rightmost *one* in row $j - d_4$ is in position $(i - d_2, j - d_4)$. Now we see that $\mathcal{R}$ has length at least $d_1 + 2d_2 + x \geq d_1 + 2d_2$. This contradicts the constraint Θ.

So we must have a run of *ones* in row $j - d_4$ in the array that is located to the right of $\mathcal{D}$ but to the left of $\mathcal{B}$. Such a run must, from Lemma 8, belong to either a $d_4 \times d_2$ block or to a non-block diagonal. The latter situation cannot arise, because such a diagonal would not be compatible with the presence of $\mathcal{B}$. It follows that the run of *ones* in row $j - d_4$ is part of a block $\mathcal{B}'$. This block is located below and to the left of $\mathcal{B}$.

We can now repeat the argument above to construct a sequence of blocks of *ones* below and to the left of $\mathcal{B}$, every row of the array intersecting with at least one block. A similar argument can also be made to show that the sequence of blocks extends upwards and to the right too.

Claim 2: The blocks constructed in Claim 1 are row leveled.

Proof of Claim 2: We have already proved that $\mathcal{B}$ and $\mathcal{B}'$ are either row leveled or partially row intersecting. Suppose the latter case holds, as illustrated in Fig. 3. Notice that $\mathcal{B}'$ is at least d_1 positions to the left of $\mathcal{B}$, so there is vertical run of *zeros* $\mathcal{V}$ just to the right of $\mathcal{B}'$ which extends at least as far as the bottom row of $\mathcal{B}'$ and as high as $\mathcal{D}$ (this run cannot be terminated by a $d_4 \times d_2$ block above $\mathcal{B}$—we have already shown that any such block must be to the right of $\mathcal{B}$).

We now calculate the length of run $\mathcal{V}$. Suppose that the top left *one* in $\mathcal{B}'$ is in position (i, j). Then the bottom *one* of diagonal $\mathcal{D}$ in column i is in position $(i, j+d_3+1+x)$ for some $x \geq 0$. By periodicity, the bottom *one* of $\mathcal{D}$ in column $i + d_2$ is in position $(i+d_2, j+d_3+1+x+d_4)$. This is the column containing $\mathcal{V}$. Now it is easy to show that $\mathcal{V}$ has length at least $d_3 + 2d_4$, violating the vertical constraint on *zeros*. The same considerations apply to any pair of consecutive blocks from our sequence.

Claim 3: The blocks constructed in Claims 1 and 2 are column leveled.

Proof of Claim 3: Consider again blocks $\mathcal{B}$ and $\mathcal{B}'$ and suppose that they are not column leveled. They are row leveled from Claim 2. Since they are not column leveled, there is a vertical run of *zeros* between them. As in Claim 2, this run violates the constraint Θ. The same argument can be deployed for any pair of blocks in the sequence.

Claim 4: The blocks in Claims 1–3 form a block diagonal of *ones*.

Proof of Claim 4: We have seen in Claims 2 and 3 that the blocks are row and column leveled. We have also seen in Claim 1 that any block lies to the left of the block above it. It follows that the blocks form a right block diagonal of *ones*.

As a summary of our proof, we have shown that in any array satisfying constraint Θ, either every *one* is contained in diagonals (some of which may be block diagonals) or there are no diagonals at all and every *one* is part of a $d_4 \times d_2$ block. □

Lemma 10: The set of arrays satisfying constraint Θ and in which every *one* is contained in a diagonal has capacity zero.

Proof: Any diagonal in an array satisfying the constraint has period (d_2, d_4) and so is determined by the positions of the *ones* in any d_4 consecutive rows. Each row can be shifted by at most $2d_2 + 1$ positions relative to the previous row. (In fact, every row after the first is determined by a shift by at most d_2 positions). Since the distance between two diagonals in a row is at least d_1 and any diagonal has width d_2, there are at most $(n)/(d_1 + d_2)$ different diagonals in an $m \times n$ array. Hence, the capacity is at most

$$\lim_{m,n\to\infty} \frac{\log_2(2d_2+1)^{\frac{d_4 n}{d_1+d_2}}}{mn} = 0. \qquad \square$$

Henceforth, we assume that there are no diagonals of *ones* in our arrays satisying constraint Θ. By Lemma 9, this means that all the *ones* occur in $d_4 \times d_2$ blocks. We next consider the situation where none of these blocks is partially intersecting.

The following result is due to Kato and Zeger [10].

Proposition 11: Let d_1 and d_3 be positive integers. Then

$$C(d_1, d_1 + 1, 1, 1; d_3, d_3 + 1, 1, 1) = 0.$$

Lemma 12: The set of arrays satisfying constraint Θ and in which no two $d_4 \times d_2$ blocks of *ones* partially intersect has capacity zero.

Proof: Consider any array with these properties. By translating the array, we can assume that the lower left corner of one of the $d_4 \times d_2$ blocks of *ones* is in position $(0, 0)$. Since no two blocks partially intersect, it follows that the lower left corner of any $d_4 \times d_2$ block of *ones* must be in position (i, j) where $i \equiv 0 \bmod, d_2$ and $j \equiv 0 \bmod d_4$. For otherwise, we would need to have an infinitely long run of *zeros* in the array.

Now from the constraint Θ it can be seen that in fact any run of *zeros* horizontally is of length ud_2 or $(u+1)d_2$ for some u, and that any run of *zeros* vertically is of length vd_4 or $(v+1)d_4$ for some v. We can now replace each run of horizontal *zeros* or *ones* by a run that is a factor of d_2 shorter, and each vertical run by a run that is a factor of d_4 shorter. We obtain a new array which satisfies the constraint $\Theta' = (u, u+1, 1, 1; v, v+1, 1, 1)$. Reversing the argument, from an array satisfying constraint Θ' we can construct a unique array satisfying the conditions of the lemma. But the capacity of the constraint Θ' is zero by Proposition 11. The lemma follows. □

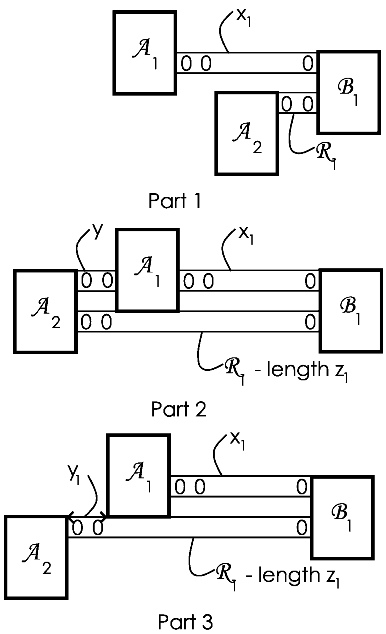


Fig. 4. Illustrating the position of blocks $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2$.

Now we can begin the final part of the proof of Theorem 7. From here on, we can assume that all the *ones* in our arrays are contained in $d_4 \times d_2$ blocks and that some pair of these blocks are partially intersecting. We also assume that there are no diagonals of *ones* in our arrays. We will prove via a sequence of lemmas that the arrangment of the blocks of *ones* is constrained to such an extent that the capacity is zero.

Let $\mathcal{A}_1$ and $\mathcal{B}_1$ be a pair of partially intersecting $d_4 \times d_2$ blocks of *ones*. We assume that these blocks are partially row intersecting, and are selected from among all such pairs of blocks of *ones* as having the smallest number of *zeros* separating them horizontally. Let x_1 denote this number. Of course $x_1 \geq d_1$. We assume that $\mathcal{A}_1$ is situated to the left of and higher than $\mathcal{B}_1$. All other configurations can be handled by very similar arguments. Fig. 4 depicts the situation we consider. This figure also shows possible placements for a block of *ones* $\mathcal{A}_2$ which must be present in order to end the run of *zeros* $\mathcal{R}_1$ that goes from the left of $\mathcal{B}_1$ in the row immediately below $\mathcal{A}_1$. Let z_1 denote the length of this run. We havethe following.

Lemma 13: With notation as above, block $\mathcal{A}_2$ is located to the left of block $\mathcal{A}_1$ and is row leveled with $\mathcal{A}_1$. Moreover, there are y_1 columns between $\mathcal{A}_1$ and $\mathcal{A}_2$, where $0 \leq y_1 \leq d_2 - 1$.

Proof: Notice that $\mathcal{A}_2$ cannot partially column intersect $\mathcal{A}_1$ because it intersects with the row immediately below $\mathcal{A}_1$. If $\mathcal{A}_2$ lay to the right of $\mathcal{A}_1$, we would have a contradiction to the minimality of x_1. So the situation in Fig. 4, Part 1 cannot arise. So $\mathcal{A}_2$ is situated to the left of $\mathcal{A}_2$. Now suppose that $\mathcal{A}_2$ partially row intersects $\mathcal{A}_1$. This situation is depicted in Fig. 4, Part 2. Let y denote the length of the run of *zeros* between $\mathcal{A}_1$ and $\mathcal{A}_2$. We have $y \geq d_1$ and $z_1 = y + d_2 + x_1 \geq 2d_1 + d_2 \geq d_1 + 2d_2$ (the last inequality following here because $d_1 \geq d_2$). This contradicts the horizontal constraint on *zeros*. It follows that $\mathcal{A}_2$ does not partially row intersect $\mathcal{A}_1$. Since it terminates the run $\mathcal{R}_1$, we must have that $\mathcal{A}_1$ and $\mathcal{A}_2$ are row leveled, as in Fig. 4, Part 3. Let y_1 denote the number of columns between the left edge of $\mathcal{A}_1$ and the right edge of $\mathcal{A}_2$. It is a simple exercise to show that $y_1 \leq d_2 - 1$.

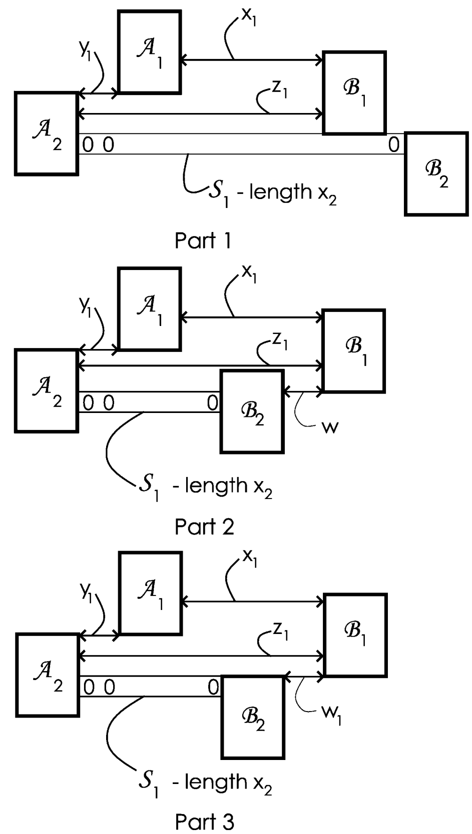


Fig. 5. Illustrating the position of blocks $\mathcal{A}_1, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_2$.

Next we consider the placement of a block $\mathcal{B}_2$ which must be present in order to end the run of *zeros* $\mathcal{S}_1$ that goes from the right of $\mathcal{A}_2$ in the row immediately below $\mathcal{B}_1$. We let $x_2 \geq d_1$ denote the length of this run. Fig. 5 depicts the situation we consider.

Lemma 14: With notation as above, block $\mathcal{B}_2$ is located to the left of block $\mathcal{B}_1$ and is row leveled with $\mathcal{B}_1$. Moreover, there are w_1 columns between $\mathcal{B}_1$ and $\mathcal{B}_2$, where $0 \leq w_1 \leq d_2 - 1$.

Proof: Arguing as before, it is easy to show that if $\mathcal{B}_2$ lies to the right of $\mathcal{B}_1$ as in Fig. 5, Part 1, then $x_2 \geq x_1 + y_1 + 2d_2 \geq d_1 + 2d_2$, contradicting the horizontal constraint on *zeros*. So $\mathcal{B}_2$ lies to the left of $\mathcal{B}_1$. Similarly, $\mathcal{B}_2$ cannot lie to the left of $\mathcal{A}_1$. $\mathcal{B}_2$ cannot partially column intersect $\mathcal{A}_1$ since it would violate the vertical constraint on the number of zeros between $\mathcal{A}_1$ and $\mathcal{B}_2$ ($d_3 \geq d_4$). Now suppose that $\mathcal{B}_2$ partially row intersects $\mathcal{B}_1$. This situation is depicted in Fig. 5, Part 2. Let w denote the length of the run of *zeros* between $\mathcal{B}_2$ and $\mathcal{B}_1$. Clearly, $w < x_1$, contradicting the minimality of x_1. It follows that $\mathcal{B}_2$ does not partially row intersect $\mathcal{B}_1$. Since $\mathcal{B}_2$ terminates the run $\mathcal{S}_1$, we must have that $\mathcal{B}_1$ and $\mathcal{B}_2$ are row leveled, as in Fig. 5, Part 3. Let w_1 denote the number of columns between the left edge of $\mathcal{B}_1$ and the right edge of $\mathcal{B}_2$. It is a simple exercise to show that $w_1 \leq d_2 - 1$. □

Lemma 15: Given the pattern of four blocks of *ones* established above, there are two infinite sequences of blocks

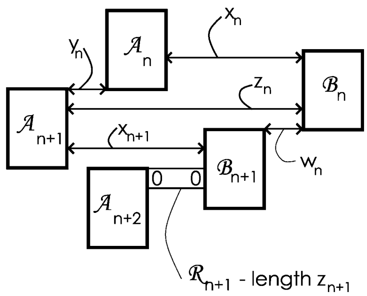


Fig. 6. Illustrating the position of blocks $\mathcal{A}_{n+1}, \mathcal{B}_{n+1}, \mathcal{A}_{n+2}$.

$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$ with the following properties:

- each pair $\mathcal{A}_i, \mathcal{A}_{i+1}$ are row leveled, $\mathcal{A}_{i+1}$ is below and to the left of $\mathcal{A}_i$, and is separated from $\mathcal{A}_i$ by y_i columns, where $0 \leq y_i \leq d_2 - 1$;
- each pair $\mathcal{B}_i, \mathcal{B}_{i+1}$ are row leveled, $\mathcal{B}_{i+1}$ is below and to the left of $\mathcal{B}_i$, and is separated from $\mathcal{B}_i$ by w_i columns, where $0 \leq w_i \leq d_2 - 1$.

Proof: The proof is by induction on n, where $n+1$ is the number of blocks placed so far in each sequence. The preceding lemmas establish the case $n = 1$. Suppose that the conditions in the lemma hold for all i up to n. We show that blocks $\mathcal{A}_{n+2}$ and $\mathcal{B}_{n+2}$ must be placed so that the conditions of the lemma hold for $i = n+1$ as well.

Referring to Fig. 6, we first consider how a block of *ones* $\mathcal{A}_{n+2}$ must be placed in order to end the run of *zeros* $\mathcal{R}_{n+1}$ that goes from the left of $\mathcal{B}_{n+1}$ in the row immediately below $\mathcal{A}_{n+1}$. Let z_{n+1} denote the length of this run. We have $z_{n+1} \geq d_1$. Let x_{n+1} denote the length of the run of *zeros* between $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$.

Notice that $\mathcal{A}_{n+2}$ cannot partially column intersect $\mathcal{A}_{n+1}$ because it intersects with the row immediately below $\mathcal{A}_{n+1}$. Suppose $\mathcal{A}_{n+2}$ lies to the right of $\mathcal{A}_{n+1}$, as in Fig. 6. From the figure, we have $d_1 \leq x_{n+1} \leq z_n - d_2$. Also,

$$\begin{aligned} d_1 \leq z_{n+1} &\leq x_{n+1} - d_2 \\ &\leq z_n - 2d_2 \\ &\leq (d_1 + 2d_2 - 1) - 2d_2 \\ &= d_1 - 1 \end{aligned}$$

which is a contradiction. We conclude that $\mathcal{A}_{n+2}$ must be located to the left of $\mathcal{A}_{n+1}$.

Now arguing exactly as in the proof of Lemma 13, it can be shown that A_{n+2} is row leveled with A_{n+1} and is separated from $\mathcal{A}_{n+2}$ by y_{n+1} columns, where $0 \leq y_{n+1} \leq d_2 - 1$.

Finally, we consider the placement of the block $\mathcal{B}_{n+2}$ which must be present in order to end the run of *zeros* $\mathcal{S}_{n+1}$ that goes from the right of $\mathcal{A}_{n+2}$ in the row immediately below $\mathcal{B}_{n+1}$. An identical argument to that given in Lemma 14 shows that $\mathcal{B}_{n+1}, \mathcal{B}_{n+2}$ are row leveled, $\mathcal{B}_{n+2}$ is below and to the left of $\mathcal{B}_{n+1}$, and is separated from $\mathcal{B}_{n+1}$ by w_{n+1} columns, where $0 \leq w_{n+1} \leq d_2 - 1$.

This completes the induction step and with it the proof of the lemma. □

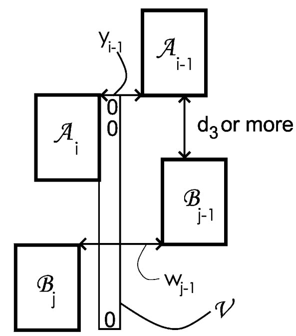


Fig. 7. Illustrating the proof of Lemma 16, Claim 1.

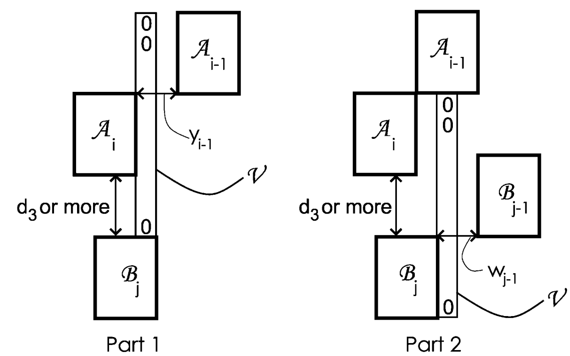


Fig. 8. Illustrating the proof of Lemma 16, Claim 2.

After this lemma, it is straightforward to see that the two sequences of blocks $\mathcal{A}_i$ and $\mathcal{B}_i$ must in fact extend in both directions, with the same properties concerning relative placement of blocks as those in Lemma 15 holding in the up and to the right direction as well. We now index the blocks in the two sequences by integers $i \in \mathbb{Z}$.

Next we focus our attention on how blocks from different sequences can be arranged vertically above one another.

Lemma 16: There exists an $r \geq 0$ such that every block $\mathcal{A}_i$ is column leveled with the block $\mathcal{B}_{i+r}$.

Proof: We prove this lemma using a sequence of claims.

Claim 1: Every column immediately to the right of a block $\mathcal{A}_i$ intersects with some block $\mathcal{B}_j$ (where j depends on i).

Proof of Claim 1: Suppose the claim is not true for some i. Then the column to the right of $\mathcal{A}_i$ must pass between two consecutive blocks, say blocks $\mathcal{B}_j$ and $\mathcal{B}_{j-1}$. This situation is shown in Fig. 7. Because $w_{j-1} \leq d_2 - 1$, we see that blocks $\mathcal{A}_{i-1}$ and $\mathcal{B}_{j-1}$ must be partially column intersecting or column matched. Now we see from the figure that the column to the right of $\mathcal{A}_i$ contains a run of *zeros* $\mathcal{V}$ of length at least $d_3 + 2d_4$, a contradiction of the vertical constraint on *zeros*.

Claim 2: When $y_{i-1} > 0$, the block $\mathcal{B}_j$ cannot partially column intersect block $\mathcal{A}_i$.

Proof of Claim 2: Suppose to the contrary that $y_{i-1} > 0$ and block $\mathcal{B}_j$ does partially column intersect block $\mathcal{A}_i$. This situation is shown in Fig. 8, Part 1. Consider the column immediately to the right of block $\mathcal{A}_i$. Notice that because $y_{i-1} \leq d_2 - 1$, there can be no *one* in this column between the top of $\mathcal{A}_{i-1}$ and the bottom of $\mathcal{A}_{i-1}$. It follows that this column contains a run of

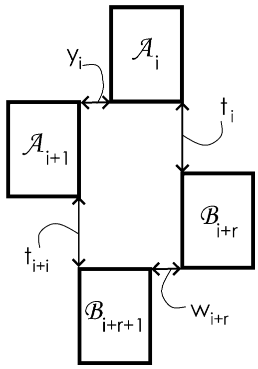


Fig. 9. The alignment of blocks $\mathcal{A}_i$, $\mathcal{A}_{i+1}$, and $\mathcal{B}_{i+r}$, $\mathcal{B}_{i+r+1}$.

zeros $\mathcal{V}$ of length at least d_3+2d_4, a contradiction of the vertical constraint on *zeros*.

Claim 3: When $y_{i-1} = 0$, the block $\mathcal{B}_j$ cannot partially column intersect block $\mathcal{A}_i$.

Proof of Claim 3: Suppose to the contrary that $y_i = 0$ and block $\mathcal{B}_j$ does partially column intersect block $\mathcal{A}_i$. This situation is shown in Fig. 8, Part 2. Suppose that $w_{j-1} > 0$. Considering the column immediately to the right of block $\mathcal{B}_j$, we get a run of *zeros* $\mathcal{V}$ contradicting the vertical constraint on *zeros*. So we must have $w_{j-1} = 0$ and we see that $\mathcal{B}_j$ and $\mathcal{B}_{j-1}$ are column leveled. Recall that $\mathcal{B}_j$ intersects with the column to the right of $\mathcal{A}_i$. This implies that $\mathcal{B}_{j-1}$ partially column intersects with $\mathcal{A}_{i-1}$. A similar argument to that just presented shows that we must have $y_{i-2} = 0$. Now we can repeat the whole argument to show that $y_k = w_k = 0$ for every $k \le i-1$. Similarly, we can prove that $y_k = w_k = 0$ for every $k > i-1$. We deduce that our array contains two block diagonals of *ones*. This contradicts our assumption that our array contains no diagonals.

Claim 4: The block $\mathcal{B}_j$ is column leveled with the block $\mathcal{A}_i$.

Proof of Claim 4: We know that $\mathcal{B}_j$ contains a column that is immediately to the right of $\mathcal{A}_i$ from Claim 1, but we have established in Claims 2 and 3 that $\mathcal{B}_j$ cannot partially column intersect $\mathcal{A}_i$. So $\mathcal{B}_j$ must be column leveled with $\mathcal{A}_i$.

The lemma itself now follows quickly. We need only show that $\mathcal{A}_i$ is always leveled with block $\mathcal{B}_{i+r}$ for some fixed $r \ge 0$. That this is the case is an easy consequence of the inequalities $y_i, w_i \le d_2 - 1$. □

Now we have established that the blocks $\mathcal{A}_i$ are column leveled with the blocks $\mathcal{B}_{i+r}$. We already know that the blocks $\mathcal{A}_i, \mathcal{A}_{i+1}$ and $\mathcal{B}_{i+r}, \mathcal{B}_{i+r+1}$ are row leveled. Let t_i denote the number of rows separating $\mathcal{A}_i$ and $\mathcal{B}_{i+r}$, as illustrated in Fig. 9. We claim that $t_i = t_j$ for every $i, j \in \mathbb{Z}$ and that $y_i = y_j = w_i = w_j$ for all $i, j \in \mathbb{Z}$. This is a straightforward consequence of the leveled nature of the various pairs of blocks. Let t denote the common value for the t_i and y the common value for the y_i, w_i. We must have $y > 0$ otherwise, there would be a diagonal of *ones* in the array. Then we can show that $0 \le t \le d_4 - 1$: this can be proved by considering the possible positions for the block of *ones* that is needed to end the run of *zeros* in the column immediately to the right of $\mathcal{A}_i$.

Now each quadruple of leveled blocks $\mathcal{A}_i, \mathcal{A}_{i+1}, \mathcal{B}_{i+r}, \mathcal{B}_{i+r+1}$ can be used to show that there must exist a pair of bi-infinite sequences of blocks extending down and to the right, the pairs beginning $\mathcal{A}_i, \mathcal{B}_{i+r}, \ldots$ and $\mathcal{A}_{i+1}, \mathcal{B}_{i+r+1}, \ldots$. The placement of blocks in each of these pairs of sequences is completely determined by the values of t and y. The proof of this for each sequence pair closely follows the proof of Lemmas 15 and 16, and uses the fact that the vertical constraints take the same form as the horizontal constraints.

Finally, we have that the capacity of the channel in the case where all *ones* are contained in $d_4 \times d_2$ blocks and where some pair of blocks is partially intersecting must be zero: any array with these properties is completely determined by the choice of y and t where $0 \le y \le d_2 - 1$ and $0 \le t \le d_4 - 1$.

This completes the proof of Theorem 7.

V. Equal Horizontal and Vertical Constraints

In this section, we examine the situation where the same constraints are applied horizontally and vertically, but where *zeros* and *ones* may be constrained differently. Our main result is as follows.

Theorem 17: Let d_1, k_1, d_2, k_2 be positive integers satisfying $d_1 \le k_1$ and $d_2 \le k_2$.

1. If $d_1 < k_1$ and $d_2 < k_2$ then $C(d_1, k_1, d_2, k_2) > 0$.
2. If $d_1 \le rd_2$ and $k_1 \ge (r+2)d_2$, for some $r \ge 1$, then $C(d_1, k_1, d_2, d_2) > 0$.
3. If $d_1 \ge d_2$ and $k_1 = d_1 + \rho$, where $0 \le \rho \le 2d_2 - 1$, then $C(d_1, k_1, d_2, d_2) = 0$.
4. If $d_2 \ge d_1$ and $k_1 \ge d_1 + 2d_2$, then $C(d_1, k_1, d_2, d_2) > 0$.
5. If $d_2 \le d_1 \le 2d_2 - 1$ and $k_1 \ge 2d_1 + d_2$, then $C(d_1, k_1, d_2, d_2) > 0$.

Proof:

1. Suppose $d_1 < k_1$ and $d_2 < k_2$. We first consider the constraint (d_1, d_1+1, d_2, d_2+1). We construct two $(d_1 + d_2 + 1) \times (d_1 + d_2 + 1)$ arrays $\mathcal{A}$ and $\mathcal{B}$. In $\mathcal{A}$, the first row consists of d_1 *zeros* followed by $d_2 + 1$ *ones*. Each of the other $d_1 + d_2$ rows is a cyclic shift by one to the right of the previous row. $\mathcal{B}$ is constructed from $\mathcal{A}$ by replacing the first *one* in the first row of $\mathcal{A}$ by a *zero*. It is easy to verify that $\mathcal{A}$ and $\mathcal{B}$ are two compatible arrays for the constraint (d_1, d_1+1, d_2, d_2+1) and hence, by Theorem 4, we have $C(d_1, d_1+1, d_2, d_2+1) > 0$. By Lemma 1 we have $C(d_1, k_1, d_2, k_2) > 0$ for $d_1 < k_1$ and $d_2 < k_2$.
2. We first study the constraint $(r, r+2, 1, 1)$. We consider the following two arrays $\mathcal{A}$ and $\mathcal{B}$. $\mathcal{A}$ is an $(r+2) \times (r+2)$ identity matrix. $\mathcal{B}$ is defined by

$$\mathcal{B} = \begin{bmatrix} I_r & X_2 \\ X_3 & X_4 \end{bmatrix}$$

where X_2 is an $r \times 2$ all-*zero* matrix, X_3 is a $2 \times r$ all-*zero* matrix, and X_4 is the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is easy to verify that $\mathcal{A}$ and $\mathcal{B}$ are compatible arrays for the constraint $(r, r+2, 1, 1)$ and hence, by Theorem 4, we have $C(r, r+2, 1, 1) > 0$. By using Lemma 3 we

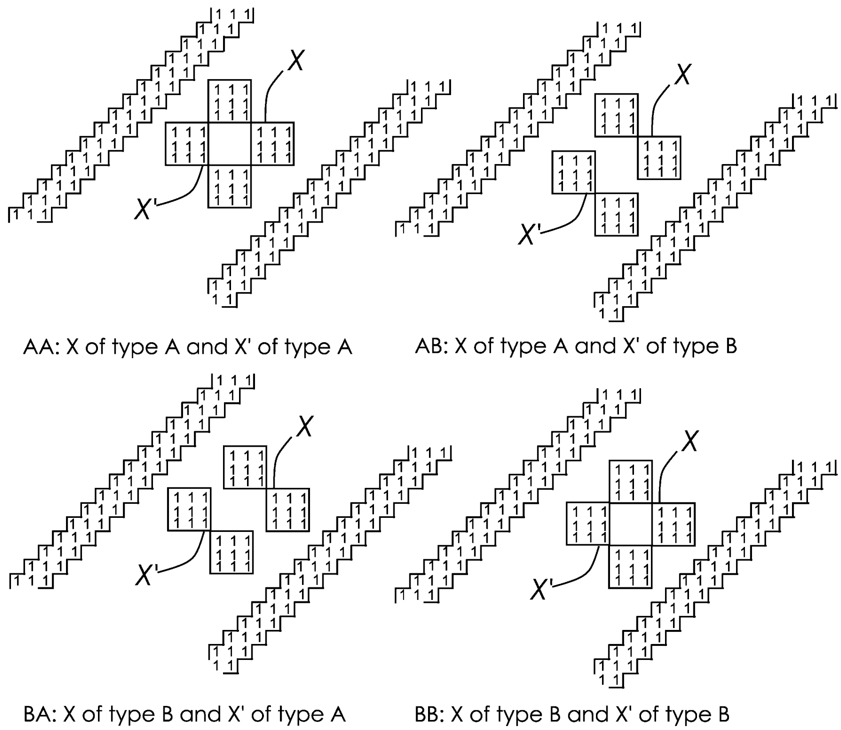


Fig. 10. Placement of *ones* in proof of Theorem 17, Part 4.

have $C(rd_2, (r+2)d_2, d_2, d_2) > 0$. By Lemma 1 we have that if $d_1 \le rd_2$, and $k_1 \ge (r+2)d_2$ for $r \ge 1$, then $C(d_1, k_1, d_2, d_2) > 0$.

3. If $d_2 = k_2, d_1 \ge d_2$ and $k_1 = d_1 + \rho$, where $0 \le \rho \le 2d_2 - 1$, then Theorem 7 gives $C(d_1, d_1+\rho, d_2, d_2) = 0$.
4. Suppose $d_2 \ge d_1$ and $k_1 = d_1 + 2d_2$. We consider the constraint (d_1, k_1, d_2, d_2) and show that its capacity is positive. Part 4 of the theorem will follow from this and Lemma 1.

In what follows, we describe only the placement of *ones* in arrays; unless otherwise stated, any other position is assumed to contain a *zero*. We begin by filling the plane with basic right diagonals of width d_2, each pair being separated by $2d_1 + 4d_2$ positions horizontally. We will show that the space between pairs of diagonals can be filled with $d_2 \times d_2$ blocks of *ones* with sufficient flexibility to ensure that the capacity is nonzero, while respecting the constraints. It will be sufficient to describe how the space between a single pair of diagonals can be filled.

We form the following array X of size $2d_2 \times 2d_2$:

$$X = \begin{bmatrix} J_{d_2} & 0_{d_2} \\ 0_{d_2} & J_{d_2} \end{bmatrix}.$$

So X consists of two blocks of *ones* joined diagonally.

We can place the array X between a pair of diagonals in two different ways, which we denote by type A and type B. In a type A placement, the upper left corner of X is separated from the left-hand diagonal of the pair by d_1 *zeros*, so that (by virtue of the spacing of diagonals) the lower right corner of X is separated from the right-hand diagonal of the pair by d_1+1 *zeros*. In a type B placement, the upper left corner of X is separated from the left-hand diagonal of the pair by d_1+1 *zeros*, so that the lower right corner of X is separated from the right-hand diagonal of the pair by d_1 *zeros*.

What we show next is that, irrepsective of whether a copy of X has been placed as type A or as type B, we can place another copy X' of X below and to the left of the first X, either as type A or as type B. As we describe this placement, the reader may find it useful to refer to Fig. 10 which illustrates our argument in the special case of the constraint $(2, 8, 3, 3)$.

Suppose X has been placed as type A. Then to place X' also as type A, we position it so that the upper right corners of the blocks of *ones* in X' are diagonally adjacent to the lower left corners of the blocks of *ones* in X (Fig. 10, Part AA). It is easy to see that X' is separated from the left-hand diagonal by d_1 *zeros*. To place X' as type B, we put it one position lower than in the previous case (Fig. 10, Part AB). Now X' is separated from the left-hand diagonal by d_1+1 *zeros*, so is of type B.

On the other hand, suppose X has been placed as type B. Then to place X' also as type B, we position it so that the upper right corners of the blocks of *ones* in X' are diagonally adjacent to the lower left corners of the blocks of *ones* in X (Fig. 10, Part BB). It is easy to see that X' is separated from the left-hand diagonal by d_1+1 *zeros*. To place X' as type A, we put it one position further to the left than in the previous case (Fig. 10, Part BA). Now X' is separated from the left-hand diagonal by d_1 *zeros*, so is of type A.

We can continue the placement of copies of X down and to the left between our two diagonals in this way. It is also easy to see how this process can be extended up and

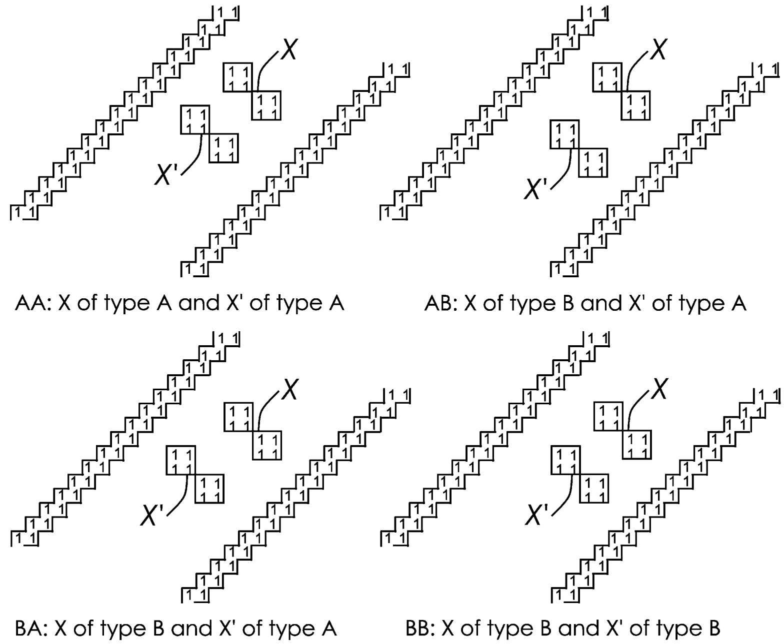


Fig. 11. Placement of *ones* in proof of Theorem 17, Part 5.

to the right of X, and to all the other pairs of diagonals. Every time we place a copy of X, we have a choice of whether to place it as type A or type B. It is a simple calculation to show that this ensures the capacity of the constraint is positive.

Finally, we need to show that the constraint (d_1, k_1, d_2, d_2) is satisfied once all the copies of X have been placed. That this is so is a routine, but tedious task. Notice however that our constraints are tight: when a type A copy of X is followed by a type B copy, a horizontal run of $d_1 + 2d_2$ *zeros* is created between the upper blocks of the two copies of X. Likewise, when a type B copy of X is followed by a type A copy, a vertical run of $d_1 + 2d_2$ *zeros* is opened up between the right-hand blocks in the two copies of X. Notice too that we have used the condition $d_2 \geq d_1$ when placing the X's: their placement so close together creates horizontal and vertical runs of *zeros* with length as small as d_2. Finally, note that the original diagonals are $2d_1 + 4d_2$ positions apart, much greater than the constraint $k_1 = d_1 + 2d_2$ on *zeros*, but the placement of X's removes all these long runs of *zeros*.

5. Suppose $d_2 \leq d_1 \leq 2d_2 - 1$ and $k_1 = 2d_1 + d_2$. We consider the constraint (d_1, k_1, d_2, d_2) and show that its capacity is positive. Part 5 of the theorem will follow from this and Lemma 1.

The remainder of our proof is very similar to part 4 above.

Again, we fill the plane with basic right diagonals, separated by $2d_1 + 4d_2$ positions. We define X in the same way as before, and consider two ways of placing X's between diagonals. As before, in a type A placement, X is situated so that the upper-left corner of X is separated from the left-hand diagonal by d_1 *zeros*. The lower-right corner of X is then separated from the right-hand diagonal of the pair by $d_1 + 1$ *zeros*. Also, as before, in a type B placement, the upper-left corner of X is separated from the left-hand diagonal by $d_1 + 1$ *zeros*, so that the lower-right corner of X is separated from the right-hand diagonal of the pair by d_1 *zeros*.

Now we consider how subsequent copies of X can be placed. Again, we can place another copy X' of X below and to the left of the first X, either as type A or as type B. As we describe this placement, the reader may find it useful to refer to Fig. 11 which illustrates our argument in the special case of the constraint $(3, 8, 2, 2)$.

Suppose X has been placed as type A. Then to place X' also as type A, we put it d_1 positions to the left and d_1 positions below X (see Fig. 11, Part AA). It is easy to see that the upper-left corner of X' is separated from the left-hand diagonal by d_1 *zeros*. To place X' as type B, we put it one position lower than in the previous case (Fig. 11, Part AB). Now the upper-left corner of X' is separated from the left-hand diagonal by $d_1 + 1$ *zeros*, so X' is of type B.

On the other hand, suppose X has been placed as type B. Then to place X' also as type B, we put it d_1 positions to the left and d_1 positions below X (Fig. 11, Part BB). To place X' as type A, we put it one position further to the left than in the previous case (Fig. 11, Part BA).

Again, there is sufficient flexibility to ensure the capacity is nonzero. Checking that the constraints are satisfied is also straightforward. Notice now that the condition $d_1 \leq 2d_2 - 1$ is needed so as to prevent a long run of *zeros* occurring between X's of type A and type B, and that runs of *zeros* of length $2d_1 + d_2$ do occur in the arrays. $\square$

The first part of Theorem 17 states that, provided $d_1 < k_1$ and $d_2 < k_2$, we have $C(d_1, k_1, d_2, k_2) > 0$. After this, and because of Lemma 2, we need only consider cases where $d_2 = k_2$. Parts

2–5 of Theorem 17 refer to special cases of this problem. It is apparent that by taking $d_2 = k_2 = 1$ in parts 2 and 3, we recover the result of [9] that $C(d, k, 1, 1) > 0$ if and only if $k > d + 1$. In fact, the "if" part of this statement is re-proved in the course of part 2 of our proof using compatible arrays. But note that we do not prove the "only if" part here. Rather, we have assumed a stronger result of [10] in the proof of Theorem 7.

It is interesting to explicitly evaluate for which constraints of type (d_1, k_1, d_2, k_2) the question of zero/positive capacity remains open. In view of part 1 of Theorem 17 and Lemma 2, we know that all open cases have $d_2 = k_2$. We divide into two subcases: $d_2 > d_1$ and $d_2 \leq d_1$.

In the first case, where $d_2 > d_1$, we have that the capacity is positive whenever $k_1 \geq d_1 + 2d_2$. When $k_1 < d_1 + 2d_2$, our results give no information, but we strongly suspect the capacity is zero in all these cases. The smallest open cases are for the constraints $(1, k_1, 2, 2)$ where we have $1 \leq k_1 \leq 3$. For each of these, it is easy to construct *ad hoc* arguments like those used in the proof of Theorem 7 to show that the capacity is indeed zero. General methods remain to be found.

In the second case, where $d_2 \leq d_1$, the situation is slightly more complex. First, consider the situation where $d_2 \leq d_1 \leq 2d_2 - 1$. Applying parts 3 and 5 of Theorem 17, we see that the unresolved cases are where $d_1 + 2d_2 \leq k_1 \leq 2d_1 + d_2 - 1$. For smaller k_1, the capacity is zero, for larger k_1, it is positive. Notice also that when $d_2 = d_1$, this range is actually empty, so the capacity question is completely resolved in this situation. We have the following.

Corollary 18: $C(d_1, k_1, d_1, d_1) > 0$ if and only if $k_1 \geq 3d_1$.

But when $d_2 < d_1$, the range is nonempty and there are unresolved cases. Of course, the constraint $(d_1, k_1, 1, 1)$ is covered by the results of [9]. The smallest unresolved case is the constraint $(3, 7, 2, 2)$. Again, we believe that the capacity is zero in all these cases.

Second, and finally, consider the situation where $d_1 \geq 2d_2$. Write $d_1 = td_2 + \epsilon$ where $1 \leq \epsilon \leq d_2$, so that $t = \lceil d_1/d_2 \rceil - 1$ where $t \geq 1$. Combining Theorem 17 part 2 (in the case $r = t + 1$) with part 3, we discover that the unresolved cases have

$$d_1 + 2d_2 \leq k_1 \leq \lceil d_1/d_2 \rceil d_2 + 2d_2 - 1.$$

The capacity is zero for smaller k_1 and positive for larger k_1. This range is empty (and the capacity question completely settled) whenever d_2 exactly divides d_1. Combining this with our previous corollary, we have the following.

Corollary 19: Suppose $s \geq 1$ and $d_2 \geq 1$. Then $C(sd_2, k_1, d_2, d_2) > 0$ if and only if $k_1 \geq (s + 2)d_2$.

Again, the range is nonempty when d_2 does not exactly divide d_1. The smallest open case is the constraint $C(5, 9, 2, 2)$. We believe the capacity to be zero in all these open cases.

VI. General Constraints

Our final results are for the most general two-dimensional runlength constraints. We begin with a zero-capacity result.

Theorem 20: Let $d_1, d_2, k_2, d_3, k_3, d_4$ be positive integers satisfying $d_2 \leq k_2, d_3 \leq k_3, d_1 \leq d_2$ and $d_4 \leq d_3$. Then $C(d_1, d_1, d_2, k_2; d_3, k_3, d_4, d_4) = 0$.

Proof: The proof consists of several steps in which we build up information about the structure of arrays satisfying the constraint $(d_1, d_1, d_2, k_2; d_3, k_3, d_4, d_4)$ where $d_1 \leq d_2$ and $d_4 \leq d_3$. We will denote this constraint by Ω. We refer the reader back to Section III for definitions.

Claim 1: Let A be an array satisfying constraint Ω. Suppose A is not composed entirely of isolated blocks of *zeros* and *ones*. Then there is at least one constant, non-block diagonal in A.

Proof of Claim 1: We illustrate the proof of this claim in Fig. 12, with A, B, C, D, E in that figure referring to the various parts of the proof below.

Because A is not composed entirely from isolated blocks, A must contain a 2×2 array that comprises either three *zeros* and a single *one*, or three *ones* and a single *zero*. We assume that in fact A contains the array

$$\mathcal{X} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}.$$

We will construct from this a left, nonblock diagonal of *ones* in A. The proof for the other seven possibilites are very similar to this case. By shifting the array if necessary and applying the vertical constraint on *ones*, we can assume that $\mathcal{X}$ is located so that there are runs of d_4 *ones* in the sets of positions

$$\{(0, j) \mid 0 \leq j \leq d_4 - 1\} \quad \text{and} \quad \{(1, j) \mid r \leq j \leq r + d_4 - 1\}$$

where $-d_4 < r < 0$. Then there are *zeros* in positions $(0, -1), (0, d_4), (1, r - 1)$ and $(1, r + d_4)$. We let s denote the largest integer such that there are *ones* in all positions

$$\{(i, j) \mid 1 \leq i \leq s, r \leq j \leq r + d_4 - 1\}.$$

We have $d_1 \leq s \leq k_2 - 1$. There are *zeros* in all positions $(i, r + d_4)$ with $1 \leq i \leq s$ and in all positions $(i, r - 1)$ with $1 \leq i \leq s$.

A. There are *zeros* in positions $(0, r)$ and $(0, r - 1)$ of the array. Otherwise, if either of these positions were filled with *ones*, there would be a vertical run of *zeros* beginning in position $(0, -1)$ of length less than $d_3 \leq d_4$, a contradiction of the vertical constraint on *zeros*.
B. Position $(s + 1, r + d_4)$ contains a *zero*. For assume the contrary. Since there is a *one* in position $(0, r + d_4)$ and *zeros* in positions $(i, r + d_4)$ for all $1 \leq i \leq s$, we would have a run of s *zeros* in row $r + d_4$ of the array, and hence we would have that $s = d_1$. On the other hand, there are *zeros* in positions $(i, r - 1)$ for $0 \leq i \leq s$ which implies that $d_1 \geq s + 1$. This gives us a contradiction.
C. There is a *one* in position $(s + 1, r)$. For, going to the left from this position along row r are s *ones* followed by a *zero* (in position $(0, r)$). But we already have a run of at least $s + 1$ *zeros* in row $r + d_4$, implying $d_1 \geq s + 1$. However, $d_2 \geq d_1$, so we must have at least $s + 1$ *ones* in the run in row r. So position $(s + 1, r)$ must contain a *one*.
D. There is a *zero* in position $(s + 1, r + d_4 - 1)$. For suppose not. This position contains a *one*. We already know that the position $(s + 1, r + d_4)$ just above it contains a *zero*.

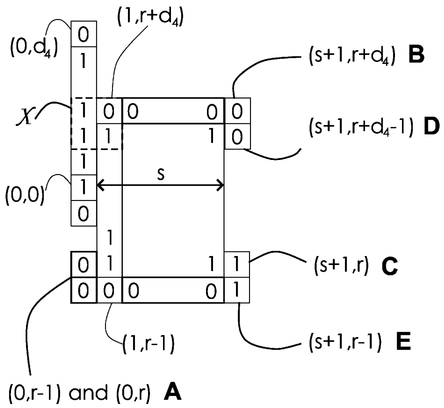


Fig. 12. Illustrating the proof of Theorem 20, Claim 1, Parts A–E.

So the *one* in position $(s+1, r+d_4-1)$ would initiate a vertical run of d_4 *ones*, ending with the *one* at position $(s+1, r)$. This would contradict the maximality of s.

E. There is a *one* in position $(s+1, r-1)$. This is the case because column $s+1$ must contain a vertical run of *ones* of length d_4 which contains the *one* in position $(s+1, r)$, but this run cannot extend higher than position $(s+1, r+d_4-2)$ because there is a *zero* in position $(s+1, r+d_4-1)$. So the run of *ones* must include position $(s+1, r-1)$.

Now looking at Fig. 12, we see that columns s and $s+1$ have the same configuration as columns 0 and 1 did before. We can therefore repeatedly apply the same argument as above to deduce that there is a left, non-block diagonal of *ones* in the array. Note, that the argument should be applied repeatedly in both directions.

Claim 2: Suppose A contains a constant diagonal and satisfies the constraint Ω. Then the whole of A is determined by this diagonal.

Proof of Claim 2: Suppose, without loss of generality, that the diagonal is a diagonal of *ones*. From the rightmost or leftmost *ones* of any row of the diagonal there must be horizontal runs of exactly d_1 *zeros*. This creates two adjacent diagonals of *zeros* of width d_1. Now from the top-most or bottom-most *zero* in any column of these diagonals, there is a vertical run of exactly d_4 *ones*. This creates two diagonals of *ones* of height d_4. Continuing in this way, we see that the whole array is determined.

Claim 3: Let A be an array satisfying constraint Ω in which every *zero* or *one* lies in an isolated block. Then the array contains a constant block diagonal.

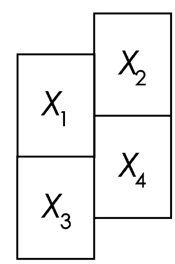
Proof of Claim 3: Consider an isolated block $\mathcal{B}$ of *ones* in the array. Each column in this block must contain exactly d_4 *ones*, and the block must contain at least d_2 columns. Next, consider the horizontal run of *zeros* that must appear directly above this block. This run has length exactly d_1. But this run must terminate all the runs of *ones* constituted by columns of $\mathcal{B}$, of which there are at least $d_2 \geq d_1$. Hence, we must have $d_2 = d_1$ and $\mathcal{B}$ has d_1 columns. Thus, $\mathcal{B}$ (and indeed any isolated block of *ones* in the array) is of size $d_4 \times d_1$. It also follows that the run of d_1 *zeros* exactly covers the columns of our block. This means that there must be a block of *ones* $\mathcal{B}'$ above and to the right of $\mathcal{B}$ and diagonally adjacent to $\mathcal{B}$. It is now easy to see how a block diagonal of *ones* is forced in the array.

Claim 4: The capacity of a channel with the constraint Ω is 0.

Proof of Claim 4: We have seen in Claims 1–3 that any array satisfying the constraint must contain a constant diagonal (either non-block or block) and that the entire array is determined by this diagonal. That the capacity of the constraint is zero now follows easily. For suppose, without loss of generality, that we have a diagonal of *ones*. Then row j of the diagonal contains at most k_2 *ones* and the position of these *ones* is determined relative to row $j-1$ by one of at most k_2+1 shifts. □

Now we present a theorem which shows that the above zero-capacity result is tight in some cases. In particular, we show that if the condition $d_4 \leq d_3$ in Theorem 20 is replaced by the condition $d_4 > d_3$, then the capacity can be positive. Before giving the theorem, we need to present a lemma which generalizes Theorem 4.

Lemma 21: Suppose $\mathcal{A}$ and $\mathcal{B}$ are two $s \times t$ $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$-constrained arrays. Consider the 16 nonsquare arrays of the form



where $X_i \in \{\mathcal{A}, \mathcal{B}\}$ and the second set of t columns are shifted up by some fixed number of positions z relative to the first set of t columns, $0 \leq z < s$. Suppose that these arrays all satisfy the $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$ constraint (except possibly at the edges of the arrays). Then

$$(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4) > 0.$$

Proof: We can fill the plane using arrays $\mathcal{A}$ and $\mathcal{B}$ as follows. First, vertically stack copies of $\mathcal{A}$ and $\mathcal{B}$ arbitrarily to form columns $\mathcal{C}_i, i \in \mathbb{Z}$ of width t. Then, for every $i \in \mathbb{Z}$, place column $\mathcal{C}_{i+1}$ to the right of column $\mathcal{C}_i$, but shifting $\mathcal{C}_{i+1}$ up by z positions relative to $\mathcal{C}_i$. From the stated properties of the arrays $\mathcal{A}, \mathcal{B}$, the set of arrays so constructed will satisfy the constraint $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$. It is clear that there is sufficient flexibility in the construction of the columns to ensure that the capacity of the constraint is positive. □

Theorem 22: Let d_1, d_2, k_2, d_3, d_4 be positive integers with $d_2 \leq d_1, k_2 \geq (2 + \lfloor (2d_4)/(d_3+1) \rfloor)d_1$ and $d_4 > d_3$. Then

$$C(d_1, d_1, d_2, k_2; d_3, d_3+1, d_4, d_4) > 0.$$

Proof: We construct a pair of arrays $\mathcal{A}, \mathcal{B}$ which can be used in Lemma 21. We only sketch the construction, and leave verification of the details to the reader.

Let $s = 2d_3 + 2d_4 + 1$ and $t' = 2 + \lceil (2d_4)/(d_3+1) \rceil$. Our arrays $\mathcal{A}$ and $\mathcal{B}$ will be of size $s \times t'd_1$. We define them as follows.

Let Y_0 be an array of *zeros* of size $d_3 \times d_1$, Y_0' an array of *zeros* of size $(d_3+1) \times d_1$ and Y_1 an array of *ones* of size $d_4 \times d_1$. Let $\mathcal{C}, \mathcal{D}$ be the arrays of size $s \times d_1$ given by

$$\mathcal{C} = \begin{bmatrix} Y_1 \\ Y_0' \\ Y_1 \\ Y_0 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} Y_1 \\ Y_0 \\ Y_1 \\ Y_0' \end{bmatrix}.$$

Let $\mathcal{C}(x)$ denote the $s \times d_1$ array obtained by cyclically shifting all the columns of $\mathcal{C}$ up by x positions, and likewise for $\mathcal{D}(x)$.

Then we define

$$\mathcal{A} = [\mathcal{C}\mathcal{D}(d_3)\mathcal{C}(d_3 + d_3 + 1)\mathcal{C}(d_3 + 2(d_3+1) \cdots \mathcal{C}(d_3 + (t-2)(d_3+1))]$$

$$\mathcal{B} = [\mathcal{D}\mathcal{C}(d_3+1)\mathcal{C}(d_3 + d_3 + 1)\mathcal{C}(d_3 + 2(d_3+1)) \cdots \mathcal{C}(d_3 + (t-2)(d_3+1))].$$

It is easy to check that $\mathcal{A}$ and $\mathcal{B}$ are suitable for use in Lemma 21 with $k_1 = d_1, k_2 = (2 + \lfloor (2d_4)/(d_3+1) \rfloor) d_1, k_3 = d_3 + 1$, $k_4 = d_4, s = 2d_3 + 2d_4 + 1, t = t'd_1$ and $z = d_3 + (t-1) \cdot (d_3+1) \bmod s$. So now we apply Lemma 21 to this pair of arrays to obtain the result. □

We illustrate the theorem with an example.

Example 23: Consider the constraint $(1,1,1,5;2,3,5,5)$. This constraint satisfies the conditions of Theorem 22. We have

$$\mathcal{A} = \begin{bmatrix} & & 0 & & & 0 \\ & & 0 & & 0 & \\ & & & & 0 & \\ & 0 & & & 0 & \\ & 0 & & 0 & & \\ 0 & & & 0 & & \\ 0 & & & 0 & & 0 \\ 0 & & 0 & & & 0 \\ & & 0 & & & \\ & & 0 & & 0 & \\ & 0 & & & 0 & \\ & 0 & & & & \\ & 0 & & 0 & & \\ 0 & & & 0 & & 0 \\ 0 & & & & & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} & & 0 & & & 0 \\ & & 0 & & 0 & \\ & 0 & & & 0 & \\ & 0 & & & 0 & \\ & 0 & & 0 & & \\ 0 & & & 0 & & \\ 0 & & & 0 & & 0 \\ & & 0 & & & 0 \\ & & 0 & & & \\ & & 0 & & 0 & \\ & 0 & & & 0 & \\ & 0 & & & & \\ 0 & & & 0 & & \\ 0 & & & 0 & & 0 \\ 0 & & & & & 0 \end{bmatrix}.$$

Notice that for clarity, we have only included the positions containing *zeros* in these arrays. All vertical runs of *ones* are of length 5. We take $z = 2$ in Lemma 21.

The parameters resulting from the construction in the proof of the preceding theorem can be improved in some cases; we omit the details.

Finally, for given d_1, d_2, and d_3, where $d_3 \le d_1$, we have found some less interesting constructions which show that

$$C(d_1 - 1, d_1, d_2, d_2 + 1; d_3 - 1, d_3, d_4, d_4 + 1) > 0$$

for some values of d_4 that depend on d_1, d_2, d_3. The reader can easily find these constructions for himself.

VII. Conclusion and Open Problems

In this paper, we have initiated the study of completely general two-dimensional runlength constraints. We have determined, for various values, whether the capacity of the $(d_1, k_1, d_2, k_2; d_3, k_3, d_4, k_4)$ runlength constraint is zero or positive. Given the difficulty in handling eight parameters simultaneously, we have focussed on the following three cases (which can, of course, be expanded using Lemma 1).

Case 1: $d_1 = d_2, k_1 = k_2, d_3 = d_4, k_3 = k_4$ (equal constraints on zeros and ones).

This case is completely solved by Theorem 6, in which we proved that $C(d_1, k_1; d_3, k_3) > 0$ if and only if $k_1 > d_1$ and $k_3 > d_3$.

Case 2: $d_1 = d_3, k_1 = k_3, d_2 = d_4, k_2 = k_4$ (equal constraints horizontally and vertically).

This case is covered in Theorem 17. By Lemma 2, it is sufficient to consider cases where either $k_1 > d_1$ and $k_2 > d_2$, or where $d_2 = k_2$.

- If $k_1 > d_1$ and $k_2 > d_2$ then $C(d_1, k_1, d_2, k_2) > 0$ (Theorem 17, part 1).
- If $k_2 = d_2$ then we distinguish between two cases:
  - — $d_1 \ge d_2$. We distinguish between two subcases:
    - * If $d_1 \le k_1 \le d_1 + 2d_2 - 1$ then $C(d_1, k_1, d_2, k_2) = 0$ (Theorem 17, part 3).
    - * If $k_1 \ge d_1 + 2d_2$ we distinguish between several further subcases:
      - . If $d_1 = rd_2$ and $k_1 \ge d_1 + 2d_2 = (r+2)d_2$, for some $r \ge 1$, then $C(d_1, k_1, d_2, d_2) > 0$ (Theorem 17, part 2).
      - . If $d_2 < d_1 \le d_2 + (d_2)/(2)$ and $k_1 \ge 4d_2 \ge 2d_1 + d_2$ then $C(d_1, k_1, d_2, d_2) > 0$ (Theorem 17, part 2).
      - . If $d_2 + (d_2)/(2) \le d_1 < 2d_2$ and $k_1 \ge 2d_1 + d_2$ then $C(d_1, k_1, d_2, d_2) > 0$ (Theorem 17, part 5).
      - . If $(r-1)d_2 < d_1 < rd_2$ and $k_1 \ge (r+2)d_2$, for some $r \ge 3$, then $C(d_1, k_1, d_2, d_2) > 0$ (Theorem 17, part 2).
      - . If $d_2 < d_1 \le d_2 + (d_2)/(2)$ and $k_1 < 4d_2$ then whether $C(d_1, k_1, d_2, d_2)$ is zero or positive is NOT KNOWN.
      - . If $d_2 + (d_2)/(2) \le d_1 < 2d_2$ and $k_1 < 2d_1 + d_2$ then whether $C(d_1, k_1, d_2, d_2)$ is zero or positive is NOT KNOWN.
      - . If $(r-1)d_2 < d_1 < rd_2$ and $k_1 < (r+2)d_2$, for some $r \ge 3$, then whether $C(d_1, k_1, d_2, d_2)$ is zero or positive is NOT KNOWN.
  - — $d_1 \le d_2$. We distinguish between two subcases:
    - * If $k_1 \ge d_1 + 2d_2$, then $C(d_1, k_1, d_2, d_2) > 0$ (Theorem 17, part 4).
    - * If $d_1 < k_1 < d_1 + 2d_2$, then whether $C(d_1, k_1, d_2, d_2)$ is zero or positive is NOT KNOWN.

Note that the case-by-case analysis in Case 2 above is consistent with the analysis which follows Theorem 17.

Case 3: $d_i = k_i$ and $d_j = k_j$ for some $i \neq j$.

By Lemma 2, we need only distinguish between three cases:

- If $d_1 = k_1$ and $d_2 = k_2$ then
$$C(d_1, d_1, d_2, d_2; d_3, k_3, d_4, k_4) = 0$$
(Theorem 5).
- If $d_1 = k_1$ and $d_3 = k_3$ then
$$C(d_1, d_1 + r_1, d_2, d_2; d_3, d_3 + r_3, d_4, d_4) = 0$$
whenever $d_1 \geq d_2, d_3 \geq d_4, 0 \leq r_1 \leq 2d_2 - 1$, and $0 \leq r_3 \leq 2d_4 - 1$ (Theorem 7).
- If $d_1 = k_1$ and $d_4 = k_4$ then
—If $d_2 \leq k_2, d_3 \leq k_3, d_1 \leq d_2$ and $d_4 \leq d_3$, then
$$C(d_1, d_1, d_2, k_2; d_3, k_3, d_4, d_4) = 0$$
(Theorem 20).
—If $d_2 \leq d_1, k_2 \geq (2 + \lfloor (2d_4)/(d_3 + 1) \rfloor) d_1$, $d_3 < k_3$, and $d_4 > d_3$, then
$$C(d_1, d_1, d_2, k_2; d_3, k_3, d_4, d_4) > 0$$
(Theorem 22).

For most of the other parameters of the type considered in Case 3 and which we have not specifically mentioned in this summary, we do not currently know if the capacity is zero or positive. This determination is left as an open problem for further research. Some parameter sets can be resolved using techniques like those we have developed, but as they are minor cases we do not include them here. Likewise, there are many parameter sets of a challenging nature left open under Case 2 above.

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