

# Restricted Growth Functions

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This document is in two parts. The first part consists of some general comments. The second part presents an alternative formulation of your ideas.

## 1 Comments

I found this difficult to read. In particular, you are using (restricted growth) functions to represent partitions. Therefore, it is much simpler to use functional notation to express these ideas rather than the group notation (see below).

Definition 2 is somewhat misleading. What you mean is that  $G_1 < G_2$  if there exists an integer  $p$  such that  $g_{1p} < g_{2p}$  and  $g_{1i} = g_{2i}$ ,  $1 \leq i \leq p - 1$ . You should use brackets consistently in the inequality. That is  $g_{1(p+1)} < g_{2(p+1)}$  rather than  $g_{1(p+1)} < g_{2p+1}$ . (As noted above, it is much simpler to use two functions  $f_1$  and  $f_2$  and say that  $f_1(i) = f_2(i)$ ,  $1 \leq i \leq p$  and  $f_1(p+1) < f_2(p+1)$ .)

The algorithm could do with some comments – it took me a while to realise what it was doing. You are basically building two lists whose concatenation forms a “path” from  $G_1$  to  $G_2$ , where each element in the path is one step away from the previous one.

## 2 Alternative formulation

Let  $[n]$  denote the set  $\{1, \dots, n\}$ . A *restricted growth function* is a function  $f : [n] \rightarrow [n]$  such that<sup>1</sup>

$$f(1) = 1 \tag{1}$$

$$f(i+1) \leq \max\{f(1), \dots, f(i)\} + 1. \tag{2}$$

Note that there is a one-to-one correspondence between the set of restricted growth functions and the set of partitions of  $[n]$ . In particular, the restricted growth function represents a partition into  $m \leq n$  groups, where 1 by convention belongs to the first group,  $i$  belongs to the  $f(i)$ th group, and  $\max\{f(1), \dots, f(n)\} = m$ .

In the sequel we represent a restricted growth function  $f$  as a list of  $n$  integers, the interpretation being that the  $i$ th element of the list is the value of  $f(i)$ . We write  $\nabla f(i)$  to denote  $\max\{f(1), \dots, f(i)\}$  and  $\nabla f$  to denote  $\nabla f(n) = \max\{f(1), \dots, f(n)\}$ .<sup>2</sup>

**Example 2.1** *The restricted growth function  $[1, 2, 3, 4, 1, 1, 1, 1, 1]$  represents the partition  $\{\{1, 5, 6, 7, 8, 9\}, \{2\}, \{3\}, \{4\}\}$  and the restricted growth function  $[1, 2, 3, 1, 4, 1, 2, 5]$  represents the partition  $\{\{1, 4, 6\}, \{2, 7\}, \{3\}, \{5\}, \{8\}\}$ .*

<sup>1</sup>Note that we can use the following conditions: for all  $1 \leq i < j \leq n$ ,  $f(j) - f(i) \leq 1$  and  $f(1) = 1$ . This formulation perhaps more intuitively captures the notion of restricted growth.

<sup>2</sup>I have introduced this notation at the last moment. Were any of this material to be published it would be neater to use this notation as it is more compact than  $\max\{f(1), \dots, f(i)\}$ . For example, the proof of Proposition 2.3 becomes much easier to write down. Proposition 2.1 is also a late addition and illustrates the use of the notation.

**Definition 2.1** Let  $f$  and  $g$  be two restricted growth functions. We say  $f \leq g$  if  $f(i) \leq g(i)$ ,  $1 \leq i \leq n$  and define<sup>3</sup>

$$h(f, g) = \sum_{i=1}^n (f(i) - g(i)), \quad (3)$$

$$H(f, g) = \sum_{i=1}^n |f(i) - g(i)|. \quad (4)$$

**Definition 2.2** Let  $f$  and  $g$  be restricted growth functions. Define  $\overline{fg} : [n] \rightarrow [n]$ , where

$$\overline{fg}(i) = \begin{cases} f(i) & \text{if } f(i) \geq g(i), \\ g(i) & \text{otherwise.} \end{cases}$$

The function  $\overline{fg}$  simply takes the larger each of the corresponding pairs of values of  $f$  and  $g$ .

**Example 2.2** Let  $f = [1, 2, 3, 4, 1, 1, 1, 1, 1]$  and  $g = [1, 1, 1, 1, 2, 2, 2, 2, 2]$ . Then  $\overline{fg} = [1, 2, 3, 4, 2, 2, 2, 2, 2]$ .

**Proposition 2.1** Let  $f$  and  $g$  be restricted growth functions. Then  $\overline{fg}$  is a restricted growth function and  $\overline{fg} = \overline{gf}$ .

**Proof** First note that  $\overline{fg}(1) = 1$  since  $f(1) = g(1) = 1$ . Now

$$\overline{fg}(i) \leq \max \{ \nabla \overline{fg}(i-1), f(i), g(i) \} \quad (5)$$

$$\leq \max \{ \nabla f(i-1), \nabla g(i-1), f(i), g(i) \} \quad (6)$$

$$\leq \max \{ \nabla f(i-1) + 1, \nabla g(i-1) + 1 \} \quad (7)$$

$$\leq \nabla \overline{fg}(i-1) + 1 \quad (8)$$

That is,  $\overline{fg}$  is a restricted growth function. ■

**Proposition 2.2** Let  $f$  and  $g$  be restricted growth functions. Then

$$h(f, g) \leq H(f, g) \leq \frac{1}{2}n(n-1).$$

**Proof** The minimum value of  $f(i)$  is 1 and the maximum value of  $g(i)$  is  $i$ . Hence

$$H(f, g) \leq \sum_{i=2}^n i - 1 = \sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1).$$

The left hand inequality is immediate. ■

**Definition 2.3** Let  $f$  and  $g$  be restricted growth functions and let  $j$  be the smallest integer such that  $f(i) < g(i)$ ,  $2 \leq j \leq n$ . If  $j$  exists, we define the function  $\overrightarrow{fg} : [n] \rightarrow [n]$ , where

$$\overrightarrow{fg}(i) = \begin{cases} f(i) & \text{if } i \neq j, \\ f(i) + 1 & \text{otherwise.} \end{cases}$$

Informally,  $\overrightarrow{fg}$  is a restricted growth function that is one step closer to  $g$  than  $f$  is. More formally, we have the following result.

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<sup>3</sup>Note that this definition of an ordering on restricted growth functions differs from yours.

**Proposition 2.3** *Let  $f$  and  $g$  be restricted growth functions. Then  $\overrightarrow{fg}$  is a restricted growth function and*

$$H(\overrightarrow{fg}, g) = H(f, g) - 1.$$

**Proof** We first note that  $\overrightarrow{fg}(1) = 1$ . Let  $f(j) < g(j)$ . Now, since  $g(j-1) = f(j-1)$  and  $g$  is a restricted growth function, we have

$$g(j) \leq \nabla f(j-1) + 1 \quad \text{and hence} \quad f(j) \leq \nabla f(j-1). \quad (9)$$

We also have

$$\nabla \overrightarrow{fg}(j-1) = \nabla f(j-1). \quad (10)$$

By definition,

$$\overrightarrow{fg}(j) = f(j) + 1 \quad (11)$$

$$\leq \nabla f(j-1) + 1 \quad \text{by (9)} \quad (12)$$

$$= \nabla \overrightarrow{fg}(j-1) + 1 \quad \text{by (10)}. \quad (13)$$

That is,  $\overrightarrow{fg}$  is a restricted growth function. Furthermore,

$$\begin{aligned} H(\overrightarrow{fg}, g) &= \sum_{i=1}^n |\overrightarrow{fg}(i) - g(i)| \\ &= \sum_{i \neq j} |\overrightarrow{fg}(i) - g(i)| + |\overrightarrow{fg}(j) - g(j)| \\ &= \sum_{i \neq j} |f(i) - g(i)| + |f(j) + 1 - g(j)| \quad \text{by definition of } \overrightarrow{fg} \\ &= \sum_{i \neq j} |f(i) - g(i)| + g(j) - (f(j) - 1) \quad \text{since } f(j) + 1 - g(j) \leq 0 \\ &= \sum_{i \neq j} |fg(i) - g(i)| + |f(j) - g(j)| - 1 \\ &= \sum_{i=1}^n |f(i) - g(i)| - 1 \\ &= H(f, g) - 1 \end{aligned}$$

■

**Proposition 2.4** *Let  $f$  and  $g$  be restricted growth functions. Then  $\overrightarrow{fg} > f$  and  $\overrightarrow{gf} > g$ .*

**Proof** Follows trivially from the definition of  $\overrightarrow{fg}$  and  $\overrightarrow{gf}$ . ■

**Lemma 2.1** *Let  $f$  and  $g$  be two restricted growth functions such that  $H(f, g) = k$  and  $h(f, g) = m > 0$ . Let  $l = \frac{1}{2}(k - m)$ . Then there exists a sequence of  $k + 1$  restricted growth functions  $h_0, \dots, h_k$  such that*

$$h_0 = f, \quad h_k = g; \quad (14)$$

$$h_{i-1} < h_i \quad \text{and} \quad h_i = \overrightarrow{h_{i-1}g}, \quad 1 \leq i \leq l; \quad (15)$$

$$h_{k-i+1} < h_{k-i} \quad \text{and} \quad h_{k-i} = \overrightarrow{h_{k-i+1}f}, \quad 1 \leq i \leq k - l. \quad (16)$$

**Proof** To be completed ■

It can easily be shown that  $l$  is the number of steps that need to be taken from  $f$  to  $\overrightarrow{fg}$ . That is,  $h_l = \overrightarrow{fg}$ .

**Example 2.3** Let  $f = [1, 2, 3, 4, 1, 1, 1, 1, 1, ]$  and  $g = [1, 1, 1, 1, 2, 2, 2, 2, 2]$ . Then  $k = 11$ ,  $m = 1$  and  $l = 5$ . The resulting sequence of restricted growth functions is

$$\begin{aligned}
h_0 &= [1, 2, 3, 4, 1, 1, 1, 1, 1] , \\
h_1 &= [1, 2, 3, 4, 2, 1, 1, 1, 1] , \\
h_2 &= [1, 2, 3, 4, 2, 2, 1, 1, 1] , \\
h_3 &= [1, 2, 3, 4, 2, 2, 2, 1, 1] , \\
h_4 &= [1, 2, 3, 4, 2, 2, 2, 2, 1] , \\
h_5 &= [1, 2, 3, 4, 2, 2, 2, 2, 2] , \\
h_6 &= [1, 2, 3, 3, 2, 2, 2, 2, 2] , \\
h_7 &= [1, 2, 2, 3, 2, 2, 2, 2, 2] , \\
h_8 &= [1, 2, 2, 2, 2, 2, 2, 2, 2] , \\
h_9 &= [1, 1, 2, 2, 2, 2, 2, 2, 2] , \\
h_{10} &= [1, 1, 1, 2, 2, 2, 2, 2, 2] , \\
h_{11} &= [1, 1, 1, 1, 2, 2, 2, 2, 2] .
\end{aligned}$$

**Example 2.4** Let  $f = [1, 2, 3, 1, 4, 1, 2, 5]$  and  $g = [1, 2, 2, 1, 3, 3, 1, 4]$ . Then  $k = 6$ ,  $m = 2$  and  $l = 2$ . The resulting sequence of restricted growth functions is

$$\begin{aligned}
h_0 &= [1, 2, 3, 1, 4, 1, 2, 5] , \\
h_1 &= [1, 2, 3, 1, 4, 2, 2, 5] , \\
h_2 &= [1, 2, 3, 1, 4, 3, 2, 5] , \\
h_3 &= [1, 2, 3, 1, 4, 3, 2, 4] , \\
h_4 &= [1, 2, 3, 1, 4, 3, 1, 4] , \\
h_5 &= [1, 2, 3, 1, 3, 3, 1, 4] , \\
h_6 &= [1, 2, 2, 1, 3, 3, 1, 4] .
\end{aligned}$$

## 2.1 Crossover

The question that arises is whether any member of this sequence is a suitable child of the two end points (the “parents”). I would agree with you and believe that there is a strong intuitive feeling that any point on this path reflects some of the attributes of both its “parents”.

I would suggest that the most efficient way of computing a child is to calculate  $k$ ,  $m$  and  $l$ , and pick a random value  $r$  in the range  $(0, k)$  (that is excluding the end points); if  $r \leq l$  compute  $h_r$  starting from  $f$ , otherwise compute  $h_r$  starting from  $g$ . Using this method, you do not need to maintain two lists in your crossover algorithm.

## 2.2 Mutation

When we used the binary matrix representation there were four possible mutation operators, which had one of the following effects:

- an element is moved from one group to another,
- two elements are swapped from their respective groups,
- two groups are merged,
- a group is split into two groups.

It is difficult using the restricted growth function representation to see how these mutation operators can be implemented without a certain amount of overhead for chromosome repair. In

the case of the first operation above, consider the function  $[1, 2, 3, 1, 4, 1, 2, 5]$ . In order to change the group to which 2 belongs we must decide on the new group and then how that move is to be effected. Suppose that we wish to move 2 to the group containing 1, 4 and 6. Then the chromosome must have the form  $[1, 1, ?, 1, ?, 1, ?, ?]$  and  $[1, 1, 3, 1, 4, 1, 2, 5]$  must be repaired to form  $[1, 1, 2, 1, 3, 1, 2, 4]$ . However, if we wish to move 2 to the group containing 5 then the chromosome must have the form  $[1, 2, ?, 1, 2, 1, ?, ?]$  and  $[1, 2, 3, 1, 2, 1, 2, 5]$  must be repaired to form  $[1, 2, 3, 1, 2, 1, 2, 4]$ . Similar objections can be applied to the other previously proposed operations.

**Mutation procedure 1** Given a restricted growth function  $f$ , pick a random value  $r$  in  $(0, n)$  and define the mutation  $f'$  to be

$$f'(i) = \begin{cases} f(i) & \text{if } i \leq r, \\ \text{rand}(f(i-1) + 1, \nabla f(r) + 1) & \text{otherwise,} \end{cases}$$

where  $\text{rand}(x, y)$  returns a random value uniformly distributed in the range  $[1, \max\{x, y\}]$ .<sup>4</sup> A possible drawback with this implementation is that if  $k$  is small, the number of new groups created is likely to be small.

**Example 2.5** Let  $f = [1, 2, 3, 1, 4, 1, 2, 5]$  and suppose  $r = 3$ . Then  $\nabla f(r) = 3$  and a possible mutation of  $f$  is given by  $[1, 2, 3, 2, 2, 1, 2, 3]$ . That is, the partition  $\{\{1, 4, 6\}, \{2, 7\}, \{3\}, \{5\}, \{8\}\}$  is replaced by the partition  $\{\{1, 6\}, \{2, 4, 5, 7\}, \{3, 8\}\}$ .<sup>5</sup>

**Mutation procedure 2** Given a restricted growth function  $f$ , pick a random value  $r$  in  $(0, n)$  and define

$$f'(i) = \begin{cases} f(i) & \text{if } i \leq r, \\ \text{rand}'(\nabla f(r) + n - r, i - 1) & \text{otherwise,} \end{cases}$$

where  $\text{rand}'(\nabla f(r) + n - r, i - 1)$  chooses a random value  $z$  in the range  $[1, \nabla f(r) + n - r]$  and returns  $z$  if  $z \in [1, \nabla f(r)]$  and  $\nabla f(r) + 1$  otherwise.

Note that the number of groups that can potentially be created for a valid restricted growth function is  $\nabla f(r) + n - r$ , because after  $r$  choices the number of groups is  $\nabla f(r)$ . In other words, this implementation is initially more likely to create a new group and becomes progressively less likely that a new group is created. A nice feature of this implementation is that  $\nabla f(r) + n - r$  is fixed, so it is sufficient to create  $n - r$  random values using the same formula, unlike in the previous procedure. For example, if  $n = 8$ ,  $r = 3$  and  $\nabla f(r) = 2$ , we create 5 random numbers in the range  $[1, 7]$ , 1, 3, 6, 6, 2, say, then the resulting values in the mutation are 1, 3, 4, 5, 2. Another nice feature is that if  $\nabla f(r)$  is small,  $\nabla f(r) + n - r$  will be relatively large and it becomes likely that more groups will be created in the mutation process.

**Example 2.6** Let  $f = [1, 2, 3, 1, 4, 1, 2, 5]$  and suppose  $r = 3$ . Then  $\nabla f(r) = 3$  and the maximum number of groups that can be created by mutation is 8. Possible mutations of  $f$  are  $[1, 2, 3, 1, 1, 4, 4, 5]$  and  $[1, 2, 3, 4, 5, 3, 6, 2]$ .

### 3 Further work and comments

**Definition 3.1** Let  $f$  and  $g$  be restricted growth functions and let  $j$  be the largest integer such that  $f(i) > g(i)$ ,  $2 \leq j \leq n$ . If  $j$  exists, we define the function  $\overleftarrow{fg} : [n] \rightarrow [n]$ , where

$$\overleftarrow{fg}(i) = \begin{cases} f(i) & \text{if } i \neq j, \\ f(i) - 1 & \text{otherwise.} \end{cases}$$

<sup>4</sup>It may be more appropriate to choose a different distribution. This will require some experimentation.

<sup>5</sup>This mutation and the ones below were created using the random number generator in Excel.

**Proposition 3.1** *Let  $f$  and  $g$  be restricted growth functions. Then  $\overleftarrow{fg}$  is a restricted growth function.*

We can then re-cast the path from  $f$  to  $g$  as a sequence of restricted growth functions  $h_0, \dots, h_k$ , where

$$h_0 = f, \quad h_k = g; \tag{17}$$

$$h_{i-1} < h_i \quad \text{and} \quad h_i = \overrightarrow{h_{i-1}fg}, \quad 1 \leq i \leq l; \tag{18}$$

$$h_{i-1} > h_i \quad \text{and} \quad h_i = \overleftarrow{fgh_{i-1}}, \quad l+1 \leq i \leq k. \tag{19}$$