

Security Models and Proof Strategies for Plaintext-Aware Encryption

James Birkett and Alexander W. Dent

Information Security Group, Royal Holloway, University of London

Abstract. Plaintext-aware encryption is a simple concept: a public-key encryption scheme is plaintext aware if no polynomial-time algorithm can create a ciphertext without “knowing” the underlying message. However, the formal definitions of plaintext awareness are very complex. This paper analyses these formal security definitions and presents the only known viable strategy for proving a scheme is PA2 plaintext aware. At the heart of this strategy is a new notion called PA1+ plaintext awareness. This security notion conceptually sits between PA1 and PA2 plaintext awareness (although it is formally distinct from either of these notions). We show exactly how this new security notion relates to the existing notions and how it can be used to prove PA2 plaintext awareness.

1 Introduction

The accepted notion for the security of a public-key encryption scheme is IND-CCA2 security [21]. This security notion states (roughly) that an encryption scheme is secure if it is infeasible to determine any information about a message from a challenge ciphertext, except perhaps its length, even with the help of a decryption oracle which will decrypt any other ciphertext. The most common strategy to prove a scheme is secure is to provide a security reduction in which an instance of a hard mathematical problem is embedded in the public key and challenge ciphertext in such a way that (a) a polynomial-time attacker cannot distinguish this “simulated” public key/challenge ciphertext from the inputs that it would expect to receive in the IND-CCA2 security game and (b) a successful attack against the scheme implies a solution to the hard problem (with non-negligible probability).

One of the main challenges of proving the IND-CCA2 security of a public-key encryption scheme is the difficulty in simulating the decryption oracle without full knowledge of the private key of the encryption scheme. This challenge is made harder by the possibility of trivial decryption oracle queries in which an attacker submits a ciphertext to the oracle for which it already “knows” the underlying message. The simulated decryption oracle must return the correct underlying message, as otherwise the attacker would be able to distinguish the simulated decryption oracle from a real decryption oracle, despite the fact that the query does not actually help the attacker break the security of the underlying scheme.

The concept of plaintext awareness was initially developed as a tool to aid attempts to prove the IND-CCA2 security of public-key encryption schemes. The basic idea is that a scheme should be deemed plaintext aware if no polynomial-time algorithm can create a valid ciphertext without “knowing” the underlying plaintext. This mitigates the usefulness of the decryption oracle in the IND-CCA2 security model: the attacker would automatically know the decryption of any ciphertext that that attacker could submit to the decryption oracle. Thus we would hope that a scheme which is IND-CPA secure [15] and plaintext aware would be IND-CCA2 secure.

The concept of plaintext-aware encryption may be simple, but the formal implementation is complex. The problem lies in the difficulty of formally defining what is meant by the phrase ‘an attacker knows the underlying plaintext’. The initial definition of plaintext awareness, introduced by Bellare and Rogaway [5] and Bellare, Desai, Pointcheval and Rogaway [3], relies on the random oracle model. A public-key encryption scheme was said to be plaintext aware if it were possible to deduce the message corresponding to a ciphertext produced by an attacker by observing the random oracle queries made by the attacker during the construction of the ciphertext. A class of

public-key encryption schemes, which make use of a hash-based checksum, can easily be proven plaintext aware in this model. However, this definition relies on the random oracle model and has no obvious counterpart in the standard model.

A more general treatment was introduced by Bellare and Palacio [4] who introduced a definition of “knowledge” similar to the definition which appears in the analysis of zero-knowledge protocols [16]. For a zero-knowledge protocol, an algorithm \mathcal{A} is deemed to “know” a value if it could be altered to give an algorithm \mathcal{A}^* of similar complexity which could output that value. For example, assuming the hardness of discrete logarithm problem, a polynomial-time algorithm which is given a group generator g and a randomly chosen element g^x does not know x , since no polynomial-time algorithm could compute x , but would know g^{x+1} since we could alter \mathcal{A} to give an algorithm $\mathcal{A}^*(g, g^x)$ which outputs g^{x+1} . In the context of plaintext-aware encryption, Bellare and Palacio proposed that an encryption scheme be deemed plaintext-aware if (roughly) for any polynomial-time algorithm \mathcal{A} which can output ciphertexts, there exists a polynomial-time algorithm \mathcal{A}^* which will output the underlying messages of those ciphertexts given the same inputs as original algorithm. The algorithm \mathcal{A} is known as the “ciphertext creator” and the algorithm \mathcal{A}^* is known as the “plaintext extractor” for obvious reasons. It should be noted that the phrase “given the same inputs” is a very strong requirement: crucially, the plaintext extractor \mathcal{A}^* should even be given the random coins of the ciphertext creator \mathcal{A} .

From a model point of view, the approach proposed by Bellare and Palacio is very appealing. The plaintext extractor \mathcal{A}^* can be thought of as a polynomial-time “spy” which sits on the shoulder of an IND-CCA2 attacker \mathcal{A} as it creates a ciphertext, deduces the underlying message, and feeds this information back to the attacker when it makes a decryption oracle query.

This simple model for plaintext awareness is known as PA1 plaintext awareness. Unfortunately, PA1 plaintext awareness is not sufficient to allow us to prove the IND-CCA2 security of an IND-CPA secure encryption scheme. This is because the IND-CCA2 game allows the attacker to make of an extra piece of information to help them create a ciphertext: the challenge ciphertext itself. It may be possible for an attacker to use the challenge ciphertext to create a ciphertext which the plaintext extractor cannot successfully decrypt¹. In order to use the concept of plaintext awareness to prove IND-CCA2 security, Bellare and Palacio introduced a “plaintext creator” into the security model. The ciphertext creator \mathcal{A} may make of use the plaintext creator \mathcal{P} in their attempt to create a ciphertext which the plaintext extractor \mathcal{A}^* cannot decrypt. On request of the ciphertext creator, the plaintext creator will random chose a message from some polynomial-time distribution and return an encryption of that message to the ciphertext creator². This model is shown diagrammatically in Figure 1. A scheme which remains plaintext aware in the presence of any plaintext creator is deemed to be PA2 plaintext aware, and a scheme which is both PA2 plaintext aware and IND-CPA secure is necessarily IND-CCA2 secure.

The Use of Plaintext Awareness in Modern Cryptography Since the original intention of plaintext awareness was to create a mechanism by which to prove a public-key encryption scheme is IND-CCA2 secure, it can be argued that the concept has failed. In the standard model, it is very difficult to prove that scheme is plaintext aware and, as we shall see, the only known proof strategy requires that a scheme is already proven to be IND-CCA2 secure before proving it is PA2 plaintext aware. Furthermore, even for these schemes, the proofs require very strong assumptions about the difficulty of solving certain types of interactive problem. However, we would argue that the notion of plaintext awareness is intrinsically interesting as it provides insight into the reasons why certain schemes can be proven IND-CCA2 secure, while other seemingly secure schemes cannot. Moreover,

¹ This issue was not originally identified in the initial work of Bellare and Rogaway [5] and only became publicly known in the later paper of Bellare, Desai, Pointcheval and Rogaway [3]. It was this issue that meant that the original analysis of the OAEP transform [5] was insufficient to prove IND-CCA2 security. The exact security of OAEP was later established by Shoup [24] and Fujisaki *et al.* [14].

² Technically, the plaintext creator is the algorithm which produces the message that is encrypted and not the combination of the algorithm which produces the message and the encryption process which produces the ciphertext. We refer to the combination of plaintext creator and encryption algorithm as the “encryption oracle” as it returns randomly-chosen ciphertexts.

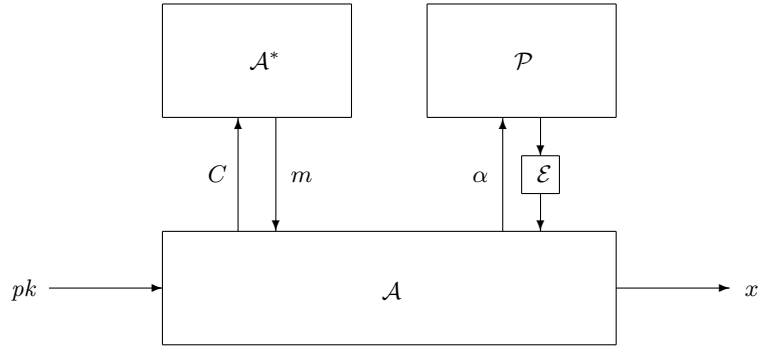


Fig. 1. The PA2 plaintext-awareness security model

there is still hope that the intrinsic problems of plaintext awareness may be solved, and it may be possible to efficiently use this technique to prove IND-CCA2 security.

The concept of plaintext awareness also has some uses independent of its potential role in proving IND-CCA2 security. For example, the use of a plaintext-aware public-key encryption schemes provide a method to reason about the actions of a participant in a protocol. Consider a protocol in which a malicious participant interacts with an honest user via a protocol which requires the malicious participant to send the honest user a ciphertext. If a plaintext-aware public-key encryption scheme is used to create the ciphertext, then we know that either the ciphertext is invalid or that the malicious participant must know the underlying message of the ciphertext. This may allow us to simulate the response of the honest user in a security reduction or to reason about the information that is leaked to the malicious participant during a protocol execution. This technique has been used to analyse deniability in the SKEME key exchange protocol [22] and in the construction of commitment schemes [29].

Our Contribution³ The core of our approach to plaintext-aware public-key encryption is the concept of PA1+ plaintext awareness. One major difference between PA1 and PA2 plaintext awareness is that, in PA1 model of plaintext awareness, when the ciphertext creator first invokes the plaintext extractor, the plaintext extractor can determine every action that the ciphertext creator will take during its entire execution. In the definition of PA2 plaintext awareness, the plaintext extractor is able to determine the actions of the ciphertext creator up to the point at which is invoked, but it cannot determine the ciphertext creator's actions after this point, as the ciphertext creator may query the encryption oracle and the plaintext extractor cannot determine the encryption oracle's responses in advance. Thus, for a probabilistic encryption scheme, the encryption oracle can actually be used to provide two services: it gives the ciphertext creator access to ciphertexts and it gives the ciphertext creator access to a source of random bits. PA1+ plaintext awareness gives the ciphertext creator access to a source of randomness, but not to a source of ciphertexts. Conceptually, the PA1+ notion of plaintext awareness sits between the PA1 and PA2 notions of plaintext awareness: the PA1+ notion of plaintext awareness gives the ciphertext creator access to a source of randomness, but does not give the ciphertext creator access to a full encryption oracle.

We show how this new notion of PA1+ plaintext awareness relates to Bellare and Palacio's existing notions of PA1 and PA2 plaintext awareness; in particular, we show that it is a weaker notion than PA2 plaintext awareness in the sense any public-key encryption scheme which is PA2 plaintext aware and IND-CPA secure is necessarily PA1+ plaintext aware. We also show that a scheme which is PA1+ plaintext aware and which presents ciphertexts which appear essentially random is necessarily PA2 plaintext aware. This gives the only known viable strategy for proving

³ This paper presents, extends, and corrects errors in [12] and [8]. We will comment on the errors in these paper as we present our (correct) results.

PA2 plaintext awareness in the standard model. We demonstrate the usefulness of this strategy by proving that the Cramer-Shoup public-key encryption scheme is PA2 plaintext aware [10].

Related Work Obviously, the work in this paper builds heavily on the standard-model definitions presented by Bellare and Palacio [4]. Bellare and Palacio also proved the fundamental result that PA2 plaintext awareness and IND-CPA security implies IND-CCA2 security. This result was improved by Teranishi and Ogata [28] who proved that PA2 plaintext awareness and one-way security implies IND-CCA2 security. Jiang and Wang investigated notions of the plaintext awareness of hybrid (KEM-DEM) encryption schemes [18] and used the proof strategy in this paper to prove that the hash-proof-based public-key encryption scheme of Kurosawa and Desmedt [19] is PA2 plaintext aware. This latter result was independently obtained by Birkett [7] who also proved that the hash-proof-based public-key encryption scheme of Cramer and Shoup [10] is PA2 plaintext aware.

2 Preliminaries

2.1 Algorithms, Assignment and Probability

We let \leftarrow denote assignment. If \mathcal{A} is a deterministic algorithm, then $y \leftarrow \mathcal{A}(x)$ denotes the assignment to y of the output of running \mathcal{A} on x . If \mathcal{A} is a probabilistic algorithm, then $y \stackrel{\$}{\leftarrow} \mathcal{A}(x)$ denotes the assignment to y of the output of running \mathcal{A} on x with a fresh set of random coins. We write $y \leftarrow \mathcal{A}(x; R)$ to denote the assignment to y of the output of running \mathcal{A} on x with the random coins R and we let $R[\mathcal{A}]$ denote the random coins of \mathcal{A} . An algorithm \mathcal{A} is (strict) polynomial-time if there exists a polynomial p such that the running time of $\mathcal{A}(x)$ is bounded by $p(|x|)$. A function $f(k, x)$ is negligible if $f(k, x) \in O(k^{-n})$ for all $x \in \{0, 1\}^*$.

The output of a probabilistic polynomial-time (PPT) algorithm \mathcal{A} with input x and access to an oracle \mathcal{O} is written $\mathcal{A}^{\mathcal{O}}(x)$. If we are describing an algorithm \mathcal{A} then we will use the phrase “Query $\beta \leftarrow \mathcal{O}(\alpha)$ ” to denote that \mathcal{A} queries the oracle \mathcal{O} on α and receives the response β . We will frequently encounter a situation where an algorithm $\mathcal{A}^{\mathcal{O}}(x)$ will be running an algorithm $\mathcal{B}^{\mathcal{O}'}(y)$ as a subroutine. In this situation \mathcal{A} will be expected to simulate the oracle \mathcal{O}' for \mathcal{B} . In order to describe this we will use a pseudocode description of the form

$$\begin{array}{l} z \stackrel{\$}{\leftarrow} \mathcal{B}^{\mathcal{O}'}(y) \\ \text{If } \mathcal{B} \text{ queries } \mathcal{O}'(\alpha) \\ \quad \text{Perform calculations} \\ \quad \text{Return } \beta \end{array}$$

to denote that \mathcal{B} is run on y , may query the “oracle” \mathcal{O}' with an input α and will receive a response β in return, and finally outputs z . We also make use of pseudocode of the form

$$\begin{array}{l} \text{Repeat } (k \text{ times}) \text{ until } E \\ \quad \text{Perform first calculations} \\ \text{Else} \\ \quad \text{Perform second calculations} \end{array}$$

to mean that the set of first calculations is repeated until either E occurs or the loop has been executed k times. If E has not occurred after k executions of the loop, then the second set of calculations is performed.

If S is a finite set, then $y \stackrel{\$}{\leftarrow} S$ denotes the assignment to y of a randomly chosen element of S . If X is a distribution over a finite set S , then $y \stackrel{\$}{\leftarrow} X$ denotes the assignment to y of an element of S chosen according to the distribution X . We refer to a list of bitstrings using the form ALIST for some identifier A. All lists are initially initialised to be empty (ALIST $\leftarrow \varepsilon$). Elements of the list are addressed as ALIST[i] with the first entry addressed as ALIST[0]; the number of the entries in the list is written as |ALIST|.

We will occasionally write

$$\Pr[\text{condition} : \text{experiment}]$$

to denote the probability that *condition* is true after executing the steps described by *experiment*. If X and Y are distributions over some common finite set S , then the statistical distance between X and Y is

$$\Delta[X, Y] = \frac{1}{2} \sum_{s \in S} |\Pr[X = s] - \Pr[Y = s]|.$$

Lemma 2.1. *Let X, Y, Z be distributions over some common finite set S . Then $\Delta[X, Z] \leq \Delta[X, Y] + \Delta[Y, Z]$. Furthermore, if \mathcal{A} is any (possibly probabilistic) algorithm, then $\Delta[\mathcal{A}(X), \mathcal{A}(Y)] \leq \Delta[X, Y]$.*

A proof of these facts can be found in, for example, Shoup [26]. The following simple lemma is not, as far as we can tell, well-known in the literature:

Lemma 2.2. *Suppose that X is the random variable describing the output of a randomised algorithm \mathcal{A} (on a finite set). Let E be some event that may occur during \mathcal{A} 's execution. Let Y be the random variable which describes the output of \mathcal{A} given that E does not occur. Then $\Delta[X, Y] \leq \Pr[E]$.*

Proof The result follows from a simple probability calculation:

$$\begin{aligned} \Delta[X, Y] &= \frac{1}{2} \sum_{s \in S} |\Pr[X = s] - \Pr[Y = s]| \\ &= \frac{1}{2} \sum_{s \in S} |\Pr[X = s | E] \Pr[E] + \Pr[X = s | \neg E] \Pr[\neg E] - \Pr[Y = s]| \\ &= \frac{1}{2} \sum_{s \in S} |\Pr[X = s | E] \Pr[E] + \Pr[Y = s](1 - \Pr[E]) - \Pr[Y = s]| \\ &= \frac{1}{2} \sum_{s \in S} |\Pr[X = s | E] \Pr[E] + \Pr[Y = s] \Pr[E]| \\ &\leq \frac{1}{2} \Pr[E] \sum_{s \in S} \Pr[X = s | E] + \frac{1}{2} \Pr[E] \sum_{s \in S} \Pr[Y = s] \\ &= \Pr[E]. \end{aligned}$$

□

These lemmas can be thought of as giving statistical versions of the “game hopping lemmas” given below.

The computational distance between X and Y for an algorithm \mathcal{A} is

$$\text{dist}_{\mathcal{A}}(X, Y) = |\Pr[\mathcal{A}(s) = 1 : s \stackrel{\$}{\leftarrow} X] - \Pr[\mathcal{A}(s) = 1 : s \stackrel{\$}{\leftarrow} Y]|.$$

X and Y are computationally indistinguishable if $\text{dist}_{\mathcal{A}}(X, Y)$ is negligible for all probabilistic polynomial-time algorithms \mathcal{A} .

Our proofs make liberal use of game hopping techniques. A game G describes the distributions of a series of inputs to an algorithm \mathcal{A} and a “win condition” for that algorithm. Let W_i denote the event that \mathcal{A} achieves the win condition. A game hop is relationship between the probability that \mathcal{A} wins game G_0 and game G_1 . We make use of three type of game hop:

1. *Bridging Steps:* The distribution of inputs to \mathcal{A} in G_0 and G_1 are identical (although the inputs may be generated in a different way). Thus, $\Pr[W_0] = \Pr[W_1]$.

2. *Distributional Steps*: If the only difference between G_0 and G_1 is that an input s to \mathcal{A} is drawn from distribution X in G_0 and from distribution Y in G_1 , then

$$|\Pr[W_0] - \Pr[W_1]| \leq \Delta[X, Y].$$

Furthermore, if \mathcal{A} is a probabilistic, polynomial-time (PPT) algorithm, and W_0 and W_1 are polynomial-time decidable events, then

$$|\Pr[W_0] - \Pr[W_1]| \leq \text{dist}_{\mathcal{A}}(X, Y).$$

3. *Small Error Steps*: If G_0 and G_1 proceed identically unless some event E occurs, then

$$|\Pr[W_0] - \Pr[W_1]| \leq \Pr[E].$$

The reader is referred to Bellare and Rogaway [6] and Shoup [25] for more information.

2.2 Public-Key Encryption Schemes

A public-key encryption scheme is a triple of PPT algorithms $(\mathcal{G}, \mathcal{E}, \mathcal{D})$. The key generation algorithm takes as input a security parameter 1^k and outputs a public/private key pair $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$. The public key implicitly defines a message space \mathcal{M} and a ciphertext space \mathcal{C} . The encryption algorithm takes a message $m \in \mathcal{M}$ and a public key pk as input, and outputs a ciphertext $C \xleftarrow{\$} \mathcal{E}(pk, m)$ in the ciphertext space \mathcal{C} . The decryption algorithm takes a ciphertext $C \in \mathcal{C}$ and a private key sk as input, and outputs either a message $m \leftarrow \mathcal{D}(sk, C)$ in the message space \mathcal{M} or the distinguished error symbol \perp . For correctness, we require that for all $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$ and messages $m \in \mathcal{M}$, we have that $\mathcal{D}(sk, \mathcal{E}(pk, m)) = m$ with probability 1.

We define five notions of security for a public-key encryption scheme: IND-ATK, LH-IND-ATK, and OW-CPA for $\text{ATK} \in \{\text{CPA}, \text{CCA2}\}$ [15, 21]. These security notions are defined via the experiments in Figure 2. For the IND/LH-IND security notions, a PPT attacker \mathcal{A} is deemed to have advantage

$$\text{Adv}_{\mathcal{A}}^{(\text{LH-})\text{IND}}(k) = |\Pr[\text{EXPT}_{\mathcal{A}}^{(\text{LH-})\text{IND}^{-1}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}}^{(\text{LH-})\text{IND}^{-0}}(k) = 1]|.$$

For the OW security notion, a PPT attacker \mathcal{A} is deemed to have advantage

$$\text{Adv}_{\mathcal{A}}^{\text{OW}}(k) = \Pr[\text{EXPT}_{\mathcal{A}}^{\text{OW}}(k) = 1].$$

A public-key encryption scheme is OW/IND/LH-IND secure if the corresponding advantage is negligible.

$\text{EXPT}_{\mathcal{A}}^{\text{IND}^{-b}}(k):$ $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$ $(m_0, m_1, \omega) \xleftarrow{\$} \mathcal{A}_1^{\mathcal{O}}(1^k, pk)$ $C^* \xleftarrow{\$} \mathcal{E}(pk, m_b)$ If $ m_0 \neq m_1 $ then $C^* \leftarrow \perp$ $b' \xleftarrow{\$} \mathcal{A}_2^{\mathcal{O}}(C^*, \omega)$ If $b' = 1$ then output 1 Else output 0	$\text{EXPT}_{\mathcal{A}}^{\text{LH-IND}^{-b}}(k):$ $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$ $(m_0, m_1, \omega) \xleftarrow{\$} \mathcal{A}_1^{\mathcal{O}}(1^k, pk)$ $C^* \xleftarrow{\$} \mathcal{E}(pk, m_b)$ $b' \xleftarrow{\$} \mathcal{A}_2^{\mathcal{O}}(C^*, \omega)$ If $b' = 1$ then output 1 Else output 0	$\text{EXPT}_{\mathcal{A}}^{\text{OW}}(k):$ $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$ $m^* \xleftarrow{\$} \mathcal{M}$ $C^* \xleftarrow{\$} \mathcal{E}(pk, m^*)$ $C \xleftarrow{\$} \mathcal{A}(1^k, pk, C^*)$ If $C = C^*$ then output 1 Else output 0
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Fig. 2. Security notions for public-key encryption: IND (left), LH-IND (center), and OW (right). In the IND and LH-IND security games, the oracle \mathcal{O} depends on ATK. If $\text{ATK} = \text{CPA}$ then the decryption oracle \mathcal{O} returns the empty string ε on all inputs. If $\text{ATK} = \text{CCA2}$ then the decryption oracle \mathcal{O} returns $\mathcal{D}(sk, C)$ on input C . The attacker \mathcal{A}_2 is forbidden from querying the decryption oracle \mathcal{O} on C^* .

The LH-IND (length-hiding indistinguishable) security notion perhaps merits a few words of explanation. A scheme is deemed to be IND secure if a ciphertext hides all partial information about a ciphertext except perhaps its length. A scheme is deemed to be LH-IND secure if it hides all partial information about a ciphertext *including* its length. Clearly, any scheme which has $\mathcal{M} = \{0, 1\}^\ell$ is LH-IND-ATK secure if and only if it is IND-ATK secure. Equally clearly, any scheme which has $\mathcal{M} = \{0, 1\}^*$ is not LH-IND-ATK secure even if it is IND-ATK secure (as one can break LH-IND-CPA security by setting m_0 to be a very short message and m_1 to be a very long message and observing the difference in ciphertext length).

2.3 Hybrid Encryption Using KEMs and DEMs

One common method to construct a practical public-key encryption scheme is to use hybrid encryption. This technique separates the public-key encryption scheme into an asymmetric key encapsulation mechanism (KEM) and a symmetric data encapsulation mechanism (DEM). Owing to the use of the symmetric DEM, the resulting public-key encryption scheme has the advantage of being able to encrypt messages of any length.

A KEM is a triple PPT algorithms $(Gen, Encap, Decap)$. The key generation algorithm takes a security parameter 1^k and outputs a public/private key pair $(pk, sk) \xleftarrow{\$} Gen(1^k)$. The public key defines a ciphertext space \mathcal{C} and a symmetric key space \mathcal{K} . The encapsulation algorithm takes a public key pk as input, and outputs a ciphertext and a symmetric key $(C, K) \xleftarrow{\$} Encap(pk)$. The decapsulation algorithm takes a private key sk and a ciphertext $C \in \mathcal{C}$, and outputs either a symmetric key $K \xleftarrow{\$} Decap(sk, C)$ or the distinguished error symbol \perp . For correctness we require that for all $(pk, sk) \xleftarrow{\$} Gen(1^k)$ and $(C, K) \xleftarrow{\$} Encap(pk)$, we have that $Decap(sk, C) = K$ with probability 1.

A DEM is defined by a pair of deterministic polynomial-time algorithms (ENC, DEC) and these algorithms implicitly define a key space \mathcal{K} , a message space \mathcal{M} , a ciphertext space \mathcal{C} . The encapsulation algorithm takes a symmetric key $K \in \mathcal{K}$ and a message $m \in \mathcal{M}$, and outputs a ciphertext $C \xleftarrow{\$} ENC_K(m)$. The decapsulation algorithm takes a symmetric key $K \in \mathcal{K}$ and a ciphertext $C \in \mathcal{C}$, and outputs either a message $m \xleftarrow{\$} DEC_K(C)$ or the distinguished error symbol \perp . For correctness we require that for all $K \in \mathcal{K}$ and $m \in \mathcal{M}$, we have that $DEC_K(ENC_K(m)) = m$.

A public-key encryption scheme may be formed from a KEM and a DEM by setting the key generation algorithm $\mathcal{G} \equiv Gen$, and computing encryption and decryption as in Figure 3. It has been shown that the combination of a sufficiently secure KEM and a sufficiently secure DEM leads to an IND-CCA2 public-key encryption scheme [10], although the exact form of that result is not relevant to this paper. However, we will note that a “sufficiently secure” DEM can be constructed through an “encrypt-then-MAC” construction using, for example, counter mode encryption with a CBC-MAC algorithm.



Fig. 3. Construction of a public-key encryption scheme from a KEM and a DEM

2.4 Simulatable Sets and Algorithms

We now introduce a novel generalisation of dense sets: simulatable sets and algorithms. Roughly speaking, a set is dense if a randomly chosen element of the set is indistinguishable from a randomly chosen element of a set $\{0, 1\}^\ell$. In essence, a set is simulatable if it is computationally

indistinguishable from the image of $\{0, 1\}^\ell$ under an invertible polynomial-time map. Since one can easily generate random elements in $\{0, 1\}^\ell$, one can also easily simulate a random element of the set by generating a random element $\{0, 1\}^\ell$ and applying the polynomial-time map (and vice versa).

We consider a family of sets indexed by a security parameter $k \in \mathbb{N}$. We may wish to simulate multiple different sets at the same security level, so we allow for the possibility of further indexing the sets as S_i for $i \in I_k$ where every $i \in I_k$ is the index of a set at security level k . Hence, we consider families of sets of the form $S = ((S_i)_{i \in I_k})_{k \in \mathbb{N}}$ or $(S_k)_{k \in \mathbb{N}}$. A simulator for a family of sets $((S_i)_{i \in I_k})_{k \in \mathbb{N}}$ is a tuple (f, f^{-1}, ℓ) where $\ell(k)$ is a polynomial and (f, f^{-1}) is a pair of PPT algorithms with the following properties:

- f is a deterministic algorithm which takes as input a security parameter 1^k , an index $i \in I_k$, and a seed $r \in \{0, 1\}^{\ell(k)}$, and outputs an element $s \in S_i$.
- f^{-1} is a probabilistic algorithm which takes as input a security parameter 1^k , an index i , and an element $s \in S$, and outputs a seed $r \in \{0, 1\}^{\ell(k)}$.
- For all $k \in \mathbb{N}$, $i \in I_k$, $s \in S_i$, we have that $f(1^k, i, f^{-1}(1^k, i, s)) = s$.

In situations where the security parameter is clear by context, we will write $f(i, r)$ and $f^{-1}(i, s)$ for $f(1^k, i, r)$ and $f^{-1}(1^k, i, s)$ respectively.

We require that $f^{-1}(1^k, i, f(1^k, i, r))$ appears to be random. Let U be the uniform distribution on $\{0, 1\}^{\ell(k)}$ and let Y be the distribution on $\{0, 1\}^{\ell(k)}$ given by $f^{-1}(1^k, i, f(1^k, i, r))$ where $r \xleftarrow{\$} \{0, 1\}^{\ell(k)}$. We define the simulator to be “statistically random-set simulatable” if $\Delta[U, Y]$ is negligible as a function of k . We define a simulator to be “computationally random-set simulatable” if the distributions U and Y are computationally indistinguishable. This can be assessed via the games in Figure 4. A PPT algorithm \mathcal{A} has advantage

$$Adv_{\mathcal{A}}^{\text{RAND}} = |\Pr[\text{EXPT}_{\mathcal{A}}^{\text{RAND-1}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}}^{\text{RAND-0}}(k) = 1]|$$

in breaking the random-set simulatable property of the simulator. A simulator is “computationally random-set simulatable” if every PPT algorithm \mathcal{A} has negligible advantage.

<p>EXPT$_{\mathcal{A}}^{\text{RAND-1}}(k)$: $b' \xleftarrow{\\$} \mathcal{A}^{\mathcal{O}_R}(1^k, i)$ If \mathcal{A} queries \mathcal{O}_R $r \xleftarrow{\\$} \{0, 1\}^\ell$ Return r Output b'</p>	<p>EXPT$_{\mathcal{A}}^{\text{RAND-0}}(k)$: $b' \xleftarrow{\\$} \mathcal{A}^{\mathcal{O}_R}(1^k, i)$ If \mathcal{A} queries \mathcal{O}_R $r \xleftarrow{\\$} \{0, 1\}^\ell$ Return $f^{-1}(1^k, i, f(1^k, i, r))$ Output b'</p>
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Fig. 4. The security games for the random-set simulatable property. It is sufficient to restrict the attacker \mathcal{A} to a single \mathcal{O}_R query as this is equivalent to the case where multiple \mathcal{O}_R queries are allowed.

We also require that $f(1^k, i, r)$ looks like a random element of the set S_i ; however, the exact form of this definition depends upon whether we talking about a simulatable set or a simulatable algorithm (of various different types). Let U be the uniform distribution on S_i and let Y be the distribution on S_i given by $f(1^k, i, r)$ for $r \xleftarrow{\$} \{0, 1\}^{\ell(k)}$. We say that the set is “statistically image-set simulatable” if $\Delta[U, Y]$ is negligible as a function of k . A set is “computationally image-set simulatable” if U and Y are computationally indistinguishable. This is defined by the games in Figure 5. A PPT attacker has advantage

$$Adv_{\mathcal{A}}^{\text{IM}}(k) = |\Pr[\text{EXPT}_{\mathcal{A}}^{\text{IM-1}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}}^{\text{IM-0}}(k) = 1]|$$

in breaking the image-set simulatable property of the simulator. A simulator is “computationally image-set simulatable” if every PPT algorithm \mathcal{A} has negligible advantage.

$\text{EXPT}_{\mathcal{A}}^{\text{IM-1}}(k)$:
 $b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_S}(1^k, i)$
 If \mathcal{A} queries \mathcal{O}_S
 $s \xleftarrow{\$} S_i$
 Return s
 Output b'

$\text{EXPT}_{\mathcal{A}}^{\text{IM-0}}(k)$:
 $b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_S}(1^k, i)$
 If \mathcal{A} queries \mathcal{O}_S
 $r \xleftarrow{\$} \{0, 1\}^{\ell(k)}$
 $s \leftarrow f(1^k, pk, r)$
 Return s
 Output b'

Fig. 5. The security games for the image-set simulatable property. It is sufficient to restrict the attacker \mathcal{A} to a single \mathcal{O}_S query as this is equivalent to the multi-query case.

Definition 2.3 (Simulatable Set). *A family of sets is statistically simulatable if there exists a simulator which is both statistically random-set and statistically image-set simulatable. A family of sets is computationally simulate if there exists a simulator which is both computationally random-set and computationally image-set simulatable.*

A simple hybrid argument/probability argument shows the following:

Lemma 2.4. *If the sets A and B are computationally/statistically simulatable, then the set $A \times B$ is computationally/statistically simulatable.*

We will require the use of public-key encryption schemes with computationally simulatable ciphertext spaces; however, since we are going to attempt to simulate ciphertexts in the presence of a decryption oracle, we require a stronger notion of simulatability than for sets. Note that we may define the ciphertext space of a public-key encryption scheme as a family of sets $((\mathcal{C}_{pk})_{pk \in PK_k})_{k \in \mathbb{N}}$ where PK_k is the set of all public keys that could be produced for the security parameter 1^k . We define ciphertext-space simulatability using the games in Figure 6. We define the attacker's advantage $\text{Adv}_{\mathcal{A}}^{\text{PKE}}(k)$ in the usual way and define a public-key encryption scheme to be computationally ciphertext-space simulatable if every PPT algorithm \mathcal{A} has negligible advantage.

$\text{EXPT}_{\mathcal{A}}^{\text{PKE-1}}(k)$:
 $(pk, sk) \xleftarrow{\$} G(1^k)$
 $b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{D}(sk, \cdot)}(1^k, pk)$
 If \mathcal{A} queries $\mathcal{O}_E(m^*)$
 $C^* \xleftarrow{\$} \mathcal{E}(pk, m^*)$
 Return m^*
 Output b'

$\text{EXPT}_{\mathcal{A}}^{\text{PKE-0}}(k)$:
 $(pk, sk) \xleftarrow{\$} G(1^k)$
 $b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{D}(sk, \cdot)}(1^k, pk)$
 If \mathcal{A} queries $\mathcal{O}_E(m^*)$
 $r \xleftarrow{\$} \{0, 1\}^{\ell(k)}$
 $C^* \leftarrow f(1^k, pk, r)$
 Return m^*
 Output b'

Fig. 6. The security games for the ciphertext-space simulatable property. The attacker may not submit any response from the encryption oracle \mathcal{O}_E to the decryption oracle $\mathcal{D}(sk, \cdot)$. It is sufficient to restrict the attacker \mathcal{A} to a single \mathcal{O}_E query as this is equivalent to the multi-query case.

Definition 2.5 (Simulate PKE). *A public-key encryption scheme is simulatable if there exists a simulator for the family $((\mathcal{C}_{pk})_{pk \in PK_k})_{k \in \mathbb{N}}$ which is computationally random-set simulatable and computationally ciphertext-space simulatable.*

The following property was first noted by Stam [27].

Lemma 2.6. *If a public-key encryption scheme is simulatable, then it is IND-CCA2 secure.*

This lemma represents one of the major challenges of plaintext awareness: all known proofs for plaintext awareness require that the public-key encryption scheme is simulatable. Thus, these proof techniques can only ever be applied to schemes *which are already known to be IND-CCA2 secure* and so plaintext awareness is not an effective tool for proving IND-CCA2 security.

We also require the notions of a simulatable KEM and simulatable DEM. These notions are very similar to that of a simulatable encryption scheme. For a simulatable KEM, we consider the family of sets which define the ciphertext space $((C_{pk})_{pk \in PK_k})_{k \in \mathbb{N}}$ as above and define ciphertext-space simulatability using the games defined in Figure 7. We define the attacker's advantage $Adv_{\mathcal{A}}^{\text{KEM}}(k)$ in the usual way and define a KEM to be computationally ciphertext-space simulatable if every PPT algorithm \mathcal{A} has negligible advantage.

$$\begin{array}{ll}
\text{EXPT}_{\mathcal{A}}^{\text{KEM-1}}(k): & \text{EXPT}_{\mathcal{A}}^{\text{KEM-0}}(k): \\
(pk, sk) \xleftarrow{\$} \text{Gen}(1^k) & (pk, sk) \xleftarrow{\$} \text{Gen}(1^k) \\
b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{D}(sk, \cdot)}(1^k, pk) & b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{D}(sk, \cdot)}(1^k, pk) \\
\text{If } \mathcal{A} \text{ queries } \mathcal{O}_E & \text{If } \mathcal{A} \text{ queries } \mathcal{O}_E \\
(C^*, K^*) \xleftarrow{\$} \text{Encap}(pk) & r \xleftarrow{\$} \{0, 1\}^{\ell(k)} \\
\text{Return } (C^*, K^*) & C^* \leftarrow f(1^k, pk, r) \\
\text{Return } b' & K^* \xleftarrow{\$} \mathcal{K} \\
& \text{Return } (C^*, K^*) \\
& \text{Return } b'
\end{array}$$

Fig. 7. The security games for the KEM ciphertext-space simulatable property. The attacker may not submit any ciphertext C^* received from the encapsulation oracle \mathcal{O}_E to the decapsulation oracle $\mathcal{D}(sk, \cdot)$. It is sufficient to restrict the attacker \mathcal{A} to a single \mathcal{O}_E query as this is equivalent to the multi-query case.

Definition 2.7 (Simulatable KEM). *A KEM is simulatable if there exists a simulator for the family $((C_{pk})_{pk \in PK_k})_{k \in \mathbb{N}}$ which is computationally random-set simulatable and computationally ciphertext-space simulatable.*

For a simulatable DEM, we consider the family of ciphertext spaces $(C_k)_{k \in \mathbb{N}}$ and define ciphertext-space simulatability via the games in Figure 8. We define the attacker's advantage $Adv_{\mathcal{A}}^{\text{DEM}}(k)$ in the usual way and define a DEM to be computationally ciphertext-space simulatable if every PPT algorithm \mathcal{A} has negligible advantage.

$$\begin{array}{ll}
\text{EXPT}_{\mathcal{A}}^{\text{DEM-1}}(k): & \text{EXPT}_{\mathcal{A}}^{\text{DEM-0}}(k): \\
K \xleftarrow{\$} \mathcal{K} & K \xleftarrow{\$} \mathcal{K} \\
b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \text{DEC}_K(\cdot)}(1^k) & b' \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \text{DEC}_K(\cdot)}(1^k) \\
\text{If } \mathcal{A} \text{ queries } \mathcal{O}_E(m^*) & \text{If } \mathcal{A} \text{ queries } \mathcal{O}_E(m^*) \\
C^* \xleftarrow{\$} \text{ENC}_K(m^*) & r \xleftarrow{\$} \{0, 1\}^{\ell(k)} \\
\text{Return } C^* & C^* \leftarrow f(1^k, r) \\
\text{Return } b' & \text{Return } C^* \\
& \text{Return } b'
\end{array}$$

Fig. 8. The security games for the DEM ciphertext-space simulatable property. The attacker may not submit any ciphertext C^* received from the encryption oracle \mathcal{O}_E to the decryption oracle $\text{DEC}_K(\cdot)$. It is sufficient to restrict the attacker \mathcal{A} to a single \mathcal{O}_E query as this is equivalent to the multi-query case.

Definition 2.8 (Simulatable DEM). *A DEM is simulatable if there exists a simulator for the family $(C)_{k \in \mathbb{N}}$ which is computationally random-set simulatable and computationally ciphertext-space simulatable.*

There is a strong similarity between the notion of simulatable DEMs and the notion of IND-R-CCA2 security for symmetric encryption schemes [23]. Using similar techniques, it is relatively easy to show that an encrypt-then-MAC DEM composed of counter mode encryption and CBC-MAC (each using an independent pseudo-random permutation) is simulatable when $\mathcal{M} = \{0, 1\}^n$ [2, 20]. The following lemma follows easily from the definitions:

Lemma 2.9. *The composition of a simulatable KEM (with a super-polynomially-sized symmetric key space) and a simulatable DEM is a simulatable hybrid encryption scheme.*

Remark 2.10. We have given the original definitions of simulatable sets/algorithms [12] in which the attacker is given access to an oracle which supplies multiple set elements/ciphertexts. A simple hybrid argument can be used to show that the definitions are equivalent to a simpler definition in which an attacker can only obtain a single set element/ciphertext. These definitions can be phrased in terms of an attacker $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ in which \mathcal{A}_1 outputs the single oracle query and \mathcal{A}_2 processes this oracle’s response. We will use both version of these definitions in this paper. To prove a scheme is simulatable we will use the single-query version of the definition; however, when making use of a simulatable set/scheme in a larger construction we will assume the multi-query definition.

3 Relations Between Notions of Plaintext Awareness

3.1 The Bellare-Palacio Notions of Plaintext Awareness

The standard-model definitions of plaintext awareness were provided by Bellare and Palacio [4]. As we have already discussed, the main idea behind the Bellare-Palacio definitions are the concepts of a ciphertext creator \mathcal{A} and a plaintext extractor \mathcal{A}^* . The ciphertext creator will be able to submit ciphertexts to a “decryption oracle”. This oracle will either be a real decryption oracle or the plaintext extractor attempting to play the part of the real decryption oracle. The plaintext extractor is given all of the inputs of the ciphertext creator, including the ciphertext creator’s random coins. Thus, the plaintext extractor can be thought of as a modified version of the ciphertext creator which outputs both ciphertexts and their underlying messages, thus realising the idea that the ciphertext creator must “know” the underlying message in the sense introduced by zero-knowledge protocols.

We define plaintext awareness using two games in Figure 9 which are played by a ciphertext creator \mathcal{A} , a plaintext extractor \mathcal{A}^* and a distinguisher algorithm D . The advantage is defined to be

$$Adv_{\mathcal{A}, \mathcal{A}^*, D}^{\text{PA1}}(k) = |\Pr[\text{EXPT}_{\mathcal{A}, D}^{\text{REAL-PA1}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, D}^{\text{FAKE-PA1}}(k) = 1]|.$$

<p>EXPT$_{\mathcal{A}, D}^{\text{REAL-PA1}}(k)$:</p> <p>$(pk, sk) \xleftarrow{\\$} \mathcal{G}(1^k)$</p> <p>$x \xleftarrow{\\$} \mathcal{A}^{\mathcal{O}_D}(1^k, pk)$</p> <p>If \mathcal{A} queries $\mathcal{O}_D(C)$:</p> <p style="padding-left: 20px;">Return $\mathcal{D}(sk, C)$</p> <p>$b \xleftarrow{\\$} D(x)$</p> <p>Output b</p>	<p>EXPT$_{\mathcal{A}, \mathcal{A}^*, D}^{\text{FAKE-PA1}}(k)$:</p> <p>$(pk, sk) \xleftarrow{\\$} \mathcal{G}(1^k)$</p> <p>$x \xleftarrow{\\$} \mathcal{A}^{\mathcal{O}_D}(1^k, pk)$</p> <p>If \mathcal{A} queries $\mathcal{O}_D(C)$:</p> <p style="padding-left: 20px;">Return $\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}])$</p> <p>$b \xleftarrow{\\$} D(x)$</p> <p>Output b</p>
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Fig. 9. The PA1 plaintext awareness security games. We assume that \mathcal{A}^* is a stateful algorithm (that retains its internal state between invocations) and recall that $R[\mathcal{A}]$ represents the random coins of \mathcal{A} .

Definition 3.1 (PA1). *A public-key encryption scheme is PA1 plaintext aware if for every PPT ciphertext creator \mathcal{A} there exists a PPT plaintext extractor \mathcal{A}^* such that for all polynomial-time distinguisher algorithms D we have that $Adv_{\mathcal{A}, \mathcal{A}^*, D}^{\text{PA1}}(k)$ is negligible.*

As we mentioned previously, PA1 plaintext awareness is not sufficient to prove IND-CCA2 security, since the model for PA1 plaintext awareness does not capture situations in which the ciphertext creator may be able to obtain ciphertexts for which they do not know the underlying message. This motivates the definition of PA2 plaintext awareness, which is defined by the two games in Figure 10. The advantage of a ciphertext creator \mathcal{A} , plaintext extractor \mathcal{A}^* , plaintext creator \mathcal{P} , and a distinguisher algorithm D is given by

$$Adv_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{PA2}}(k) = |\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-PA2}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{FAKE-PA2}}(k) = 1]|.$$

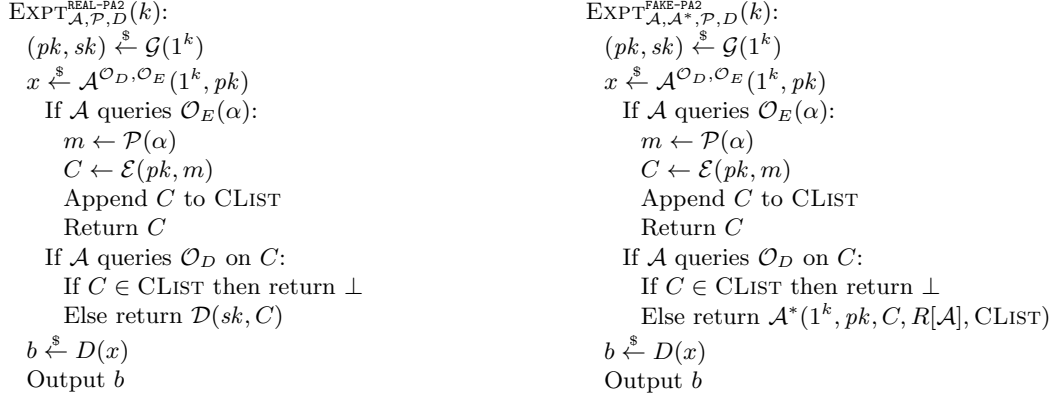


Fig. 10. The PA2 plaintext awareness security games. We assume that \mathcal{A}^* and \mathcal{P} are stateful algorithms (that retain their internal state between invocations).

Definition 3.2 (PA2). A public-key encryption scheme is PA2 plaintext aware if for every PPT ciphertext creator \mathcal{A} there exists a PPT plaintext extractor \mathcal{A}^* such that for all PPT plaintext creators \mathcal{P} and PPT distinguisher algorithms D we have that $Adv_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{PA2}}(k)$ is negligible.

The following theorems demonstrate the relationship between plaintext awareness and IND-CCA2 security:

Theorem 3.3 (Bellare-Palacio). A public-key encryption scheme that is (LH-)IND-CPA secure and PA2 plaintext aware is (LH-)IND-CCA2 secure.

Theorem 3.4 (Teranishi-Ogata). A public-key encryption scheme that is OW-CPA secure and PA2 plaintext aware is (LH-)IND-CPA secure.

Technically, both of these results are only proven for the IND security notions, but an examination of the proofs shows that neither relies on the fact that $|m_0| = |m_1|$ in the IND security game. Thus, the proofs hold for the LH-IND security notions as well. This allows us to prove the following result (which motivates our consideration of LH-IND security at all):

Corollary 3.5. A public-key encryption scheme that is IND-CPA secure, PA2 plaintext aware, and has the property that $|\mathcal{M}|$ grows faster than any polynomial is LH-IND-CPA secure.

Proof An encryption scheme which is IND-CPA scheme and which has the property that $|\mathcal{M}|$ grows faster than any polynomial is OW-CPA secure. Thus, by the Teranishi-Ogata result, the scheme is LH-IND-CPA secure. \square

Conceptually, this minor observation proves that we should only consider schemes which are LH-IND-CPA secure as candidates for PA2 plaintext awareness. Practically, this means that no scheme with $\mathcal{M} = \{0, 1\}^*$ can ever be PA2 plaintext aware, which eliminates most hybrid schemes. Interestingly, we will show that the Cramer-Shoup hybrid encryption scheme is PA2 plaintext aware if we restrict the message space to $\mathcal{M} = \{0, 1\}^k$ (and so, with the use of a suitable padding scheme, if $\mathcal{M} = \{0, 1\}^{\leq k}$).

3.2 Simplifying PA2 Plaintext Awareness

The full definition for PA2 plaintext awareness models a very general scenario: the (stateful) plaintext creator can be used to represent any number of communicating users which wish to generate any number of ciphertexts (under the influence of the ciphertext creator). This definition appears stronger than is required to prove Theorem 3.3, which only requires that we consider two plaintext creators, \mathcal{P}_0 and \mathcal{P}_1 , defined as follows⁴:

$P_0(\alpha)$: Parse α as (m_0, m_1) If parsing fails, return 0 Else return m_0	$P_1(\alpha)$: Parse α as (m_0, m_1) If parsing fails, return 0 Else return m_1
--	--

We will show that, under certain conditions, plaintext awareness with respect to these two plaintext creators is equivalent to the general PA2 plaintext awareness.

Definition 3.6 (2PA2). *A public-key encryption scheme is 2PA2 plaintext aware if for every PPT ciphertext creator \mathcal{A} there exists a PPT plaintext extractor \mathcal{A}^* such that for the plaintext creators $\mathcal{P} \in \{\mathcal{P}_0, \mathcal{P}_1\}$ and PPT distinguisher algorithms D we have that $\text{Adv}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{PA2}}(k)$ is negligible.*

We may now begin to relate 2PA2 and PA2 plaintext awareness for LH-IND-CPA and LH-IND-CCA2 schemes:

Theorem 3.7. *Let $\hat{\mathcal{P}}$ be some fixed, stateless polynomial-time plaintext creator (i.e. $\hat{\mathcal{P}}$ does not pass state between invocations). If a public-key encryption scheme Π is LH-IND-CCA2 secure and PA2 plaintext aware with respect to the plaintext creator $\hat{\mathcal{P}}$, then Π is PA2 plaintext aware.*

Proof Let $\Pi = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ and \mathcal{A} be any PA2 ciphertext creator. Then, since Π is PA2 plaintext aware for the plaintext creator $\hat{\mathcal{P}}$, there exists a plaintext extractor \mathcal{A}^* which “simulates” the decryption oracle in $\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \hat{\mathcal{P}}, D}^{\text{FAKE-PA2}}(k)$. We show that \mathcal{A}^* “simulates” the decryption oracle for any plaintext creator \mathcal{P} .

Suppose \mathcal{A} makes at most n queries to the plaintext creator oracle. Consider $\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-PA2}}(k)$ instantiated with an arbitrary PPT plaintext creator \mathcal{P} . Define the games $\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}i}(k)$ to be identical to $\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-PA2}}(k)$ except that the final i plaintext creator oracle queries are answered using the plaintext creator $\hat{\mathcal{P}}$. Hence, $\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-PA2}}(k) = \text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-0}}(k)$ and $\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-PA2}}(k) = \text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}n}(k)$. Suppose that

$$|\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-0}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}n}(k)]|$$

is non-negligible. Then, by a hybrid argument, there exists a sequence of values i_k such that

$$|\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}i_k}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}(i_k + 1)}(k)]|$$

is non-negligible. We use this to build an LH-IND-CCA2 attacker $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ as follows:

⁴ The original version of this result [8] claimed that plaintext awareness with respect to a single ciphertext creator \mathcal{P}_I , which chooses a random bit b and then consistently outputs m_b when given $\alpha = (m_0, m_1)$, was sufficient to prove IND-CCA2 security. We called this property PA2I plaintext awareness. Sadly, this result is not correct. We will sketch a counter-example here. Suppose $\Pi = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ is IND-CCA2 secure and PA2I plaintext aware. Let $\Pi' = (\mathcal{G}, \mathcal{E}', \mathcal{D}')$ be the encryption scheme with $\mathcal{E}'(pk, m) = 0 \parallel \mathcal{E}(pk, m)$ and $\mathcal{D}'(sk, \delta \parallel C) = \mathcal{D}(sk, C)$ for any $\delta \in \{0, 1\}$. This scheme is IND-CPA secure but not IND-CCA2 secure. We claim that this scheme is still PA2I plaintext aware. We may build a plaintext extractor \mathcal{A}^* for a ciphertext creator \mathcal{A} working against Π' by noting two properties of the scheme. (a) Since Π is PA2I there exists a plaintext extractor which can decrypt all decryption queries of the form $\delta \parallel C$ where $0 \parallel C \notin \text{CLIST}$. (b) If \mathcal{A} requests the decryption of $1 \parallel C$ where $0 \parallel C \in \text{CLIST}$ then the correct response must be either m_0 or m_1 , which can be determined by observing the original encryption oracle query. Furthermore, since Π is IND-CCA2 secure, no algorithm can distinguish between the case where \mathcal{A}^* returns the correct decryption of $1 \parallel C$ and the case where \mathcal{A}^* returns the message m_a (for some value d which was randomly chosen by \mathcal{A}^* during its first execution). Thus Π' is IND-CPA secure and PA2I plaintext aware, but not IND-CCA2 secure, which contradicts the claimed result of [8]

$\mathcal{B}_1^{\mathcal{D}}(1^k, pk):$ Pick $j \xleftarrow{\$} \{1, \dots, n\}$ $ctr \leftarrow 0$ $x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_D, \mathcal{O}_E}(1^k, pk)$ If \mathcal{A} queries $\mathcal{O}_D(C)$ Query $m \leftarrow \mathcal{D}(C)$ Return m If \mathcal{A} queries $\mathcal{O}_E(\alpha)$ $ctr \leftarrow ctr + 1$ If $ctr \neq j$ then return $\mathcal{E}(pk, \mathcal{P}(\alpha))$ If $ctr = j$ then $m_0 \leftarrow \mathcal{P}(\alpha)$, $m_1 \leftarrow \hat{\mathcal{P}}(\alpha)$ Pause \mathcal{A} and output (m_0, m_1)	$\mathcal{B}_2^{\mathcal{D}}(C^*):$ Resume $x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_D, \mathcal{O}_E}(1^k, pk)$ by returning C^* If \mathcal{A} queries $\mathcal{O}_D(C)$ Query $m \leftarrow \mathcal{D}(C)$ Return m If \mathcal{A} queries $\mathcal{O}_E(\alpha)$ Return $\mathcal{E}(pk, \hat{\mathcal{P}}(\alpha))$ Output $D(x)$
--	--

Thus,

$$\begin{aligned} Adv_B^{\text{LH-IND}}(k) &= |\Pr[\text{EXPT}_{\mathcal{B}}^{\text{LH-IND-1}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{B}}^{\text{LH-IND-0}}(k) = 1]| \\ &\geq \frac{1}{n} |\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}i_k}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-}(i_k+1)}(k) = 1]|. \end{aligned}$$

The fact that $(\mathcal{G}, \mathcal{E}, \mathcal{D})$ is LH-IND-CCA2 secure means that $Adv_{\mathcal{B}}^{\text{LH-IND}}(k)$ is negligible, which is a contradiction. A similar argument (based on LH-IND-CPA security) shows that

$$|\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{FAKE-PA2}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \hat{\mathcal{P}}, D}^{\text{FAKE-PA2}}(k) = 1]|$$

is negligible. However, since Π is 2PA2 plaintext aware, we have that

$$|\Pr[\text{EXPT}_{\mathcal{A}, \hat{\mathcal{P}}, D}^{\text{REAL-PA2}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \hat{\mathcal{P}}, D}^{\text{FAKE-PA2}}(k) = 1]|$$

is negligible. Thus, we conclude that

$$|\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, D}^{\text{REAL-PA2}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{FAKE-PA2}}(k) = 1]|$$

is negligible, which means that Π is PA2 plaintext aware. \square

Corollary 3.8. *A public-key encryption scheme Π which is LH-IND-CPA secure and 2PA2 plaintext aware is PA2 plaintext aware.*

Proof If Π is LH-IND-CPA secure and 2PA2 plaintext aware, then Π is LH-IND-CCA2 secure by Theorem 3.3. And so, by Theorem 3.7, it is PA2 plaintext aware. \square

3.3 PA1+ Plaintext Awareness

The crux of our approach to plaintext awareness is new a definition which we term PA1+ plaintext awareness. In the definition of PA1 plaintext awareness, when the ciphertext creator \mathcal{A} first invokes the plaintext extractor \mathcal{A}^* , the plaintext extractor can determine every action that the ciphertext creator will take during its entire execution. In PA2 plaintext awareness, the plaintext extractor is able to determine the actions of the ciphertext creator up to the point at which is invoked. It cannot determine the ciphertext creator's actions after this point, as the ciphertext creator may query the encryption oracle and the plaintext extractor cannot determine the encryption oracle's responses in advance. Thus, for a probabilistic encryption scheme, the encryption oracle can actually be used to provide two services: it gives the ciphertext creator access to ciphertexts and it gives the ciphertext creator access to a source of random bits. PA1+ plaintext awareness gives the ciphertext creator access to a source of randomness, but not to a source of ciphertexts.

Definition 3.9 (PA1+). For any definition of plaintext awareness $PA \in \{PA1/2PA2/PA2\}$, we give a new definition $PA+$ in which the ciphertext creator \mathcal{A} is given access to a randomness oracle, which takes no input and returns a randomly-chosen bit. The plaintext extractor \mathcal{A}^* is altered so that it takes a list R_{LIST} of the random bits returned by the randomness oracle as an additional input, i.e. the plaintext extractor is run as $\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], CLIST, RLIST)$.

Another interpretation of the difference between the PA and PA+ definitions relates to the way one defines a probabilistic Turing machine. If one defines a probabilistic Turing machine as a Turing machine with access to infinite random tape, then one automatically derives the PA notions of plaintext awareness. If one defines a probabilistic Turing machine as a Turing machine with access to an oracle which will return a randomly chosen bit, then one derives the PA+ notions of plaintext awareness. It is interesting that these two definitions of probabilistic Turing machine, which are usually equivalent, lead to different definitions of plaintext awareness.

It is trivial to see that $PA2+ \Rightarrow PA1+ \Rightarrow PA1$ and $PA2+ \Rightarrow PA2 \Rightarrow PA1$ (where $X \Rightarrow Y$ means that any public-key encryption scheme which satisfies notion X must also satisfy notion Y). We would hope that, for sufficiently random encryption schemes, we also have that $PA2 \Rightarrow PA2+$, and this does indeed turn out to be the case. We will also see that, for simulatable encryption schemes, $PA1+ \Rightarrow PA2+$. This is unsurprising as ciphertexts produced by a simulatable encryption scheme are indistinguishable from random bits; hence, an oracle that produces ciphertexts is equivalent to an oracle which produces random bits. It is the last relationship that gives our strategy for proving PA2 plaintext awareness. The relationship between the notions of plaintext awareness is shown in Figure 11.

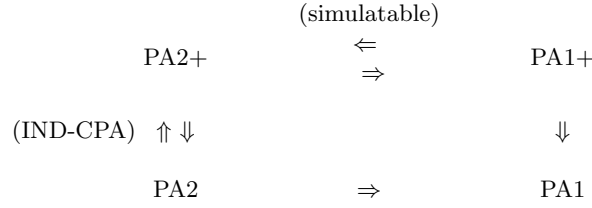


Fig. 11. The relationship between notions of plaintext awareness

In this section, we will prove that schemes which are IND-CPA and PA2 plaintext aware are necessarily PA2+ plaintext aware (and therefore trivially PA1+ plaintext aware). Our proof uses a universal hash family [9] to extract the randomness inherent in the ciphertext produced by an IND-CPA encryption scheme.

Definition 3.10 (Universal Hash Family). A family (H, K, A, B) of functions $(H_k)_{k \in K}$, where $H_k : A \rightarrow B$ for all $k \in K$, is universal if for all $x, y \in A$ satisfying $x \neq y$ we have

$$\Pr[H_k(x) = H_k(y) : k \xleftarrow{\$} K] \leq 1/|B|.$$

Definition 3.11 (Collision Probability). For a random variable x which takes values on a set X , the collision probability is defined to be

$$\kappa(x) = \sum_{y \in X} \Pr[x = y]^2.$$

Lemma 3.12. The following results will be useful in later proofs:

1. If x is a random variable on a set X , then $\max_{y \in X} \Pr[x = y] \leq \sqrt{\kappa(x)}$.
2. Let $(\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a public-key encryption scheme and M be a deterministic polynomial-time algorithm which outputs a message $M(pk) \in \mathcal{M}$. If x is the random variable on \mathcal{C} distributed according to $\mathcal{E}(pk, M(pk))$, then there exists an IND-CPA adversary \mathcal{B} such that $\kappa(x) = \text{Adv}_{\mathcal{B}}^{\text{IND}}(k)$.

The first part follows trivially from the definition of collision probability. The second part can easily be proven by noting that if $C_1, C_2 \xleftarrow{\$} \mathcal{E}(pk, M(pk))$ then the probability that $C_1 = C_2$ is $\kappa(x)$. The IND-CPA attacker \mathcal{B} is constructed by setting \mathcal{B}_1 to output $m_0 \leftarrow M(pk)$ and some arbitrary $m_1 \neq m_0$, and \mathcal{B}_2 output 0 if $C^* = \mathcal{E}(pk, M(pk))$ and 1 otherwise.

The leftover hash lemma was originally proven by Håstad *et al.* [17]. We present the version given by Shoup [26, Theorem 6.21]:

Lemma 3.13 (Leftover Hash Lemma). *Let (H, K, A, B) be a family of universal hash functions. If $k \xleftarrow{\$} K$, and x_1, \dots, x_ℓ are random variables on A which are independent of k , and $y_1, \dots, y_\ell \xleftarrow{\$} B$, then*

$$\Delta[(k, H_k(x_1), \dots, H_k(x_\ell)), (k, y_1, \dots, y_\ell)] \leq \ell \sqrt{|B| \kappa} / 2 \quad \text{where} \quad \kappa = \max_{i=1}^{\ell} \{\kappa(x_i)\}.$$

We are now in a position to state and prove the main theorem of this section:

Theorem 3.14. *If a public-key encryption scheme Π is IND-CPA secure and 2PA2 plaintext aware, then it is 2PA2+ plaintext aware.*

Proof Suppose $\Pi = (\mathcal{G}, \mathcal{E}, \mathcal{D})$. We intend to simulate the randomness oracle by hashing ciphertexts $\mathcal{E}(pk, 0)$ using a universal hash function. Let $\ell(k)$ be a (polynomial) upper-bound on the length of $\mathcal{E}(pk, 0)$ (such a bound exists as \mathcal{E} is a strict polynomial-time algorithm). Let A be the set of all strings of length at most $\ell(k)$ and $B = \{0, 1\}$. We can construct a universal hash family from A to B without computational assumptions [9], so let $(H_k)_{k \in K}$ be such a family.

Let \mathcal{A} be a 2PA2+ ciphertext creator (i.e. \mathcal{A} expects to have access to encryption, decryption, and randomness oracles). Let q_D and q_R be a bound on the number of decryption and randomness oracle queries respectively. We construct a 2PA2 ciphertext creator \mathcal{B} (i.e. \mathcal{B} has access to an encryption and decryption oracle) as in Figure 12. (Note that we may assume that there exists a polynomial-bound $t(k)$ on the number of random bits on \mathcal{A} 's initial random tape as \mathcal{A} is polynomial time.) Since $(\mathcal{G}, \mathcal{E}, \mathcal{D})$ is 2PA2 plaintext aware, there exists a plaintext extractor \mathcal{B}^* for \mathcal{B} . We use \mathcal{B}^* to construct a 2PA2+ plaintext extractor \mathcal{A}^* for \mathcal{A} as in Figure 12.

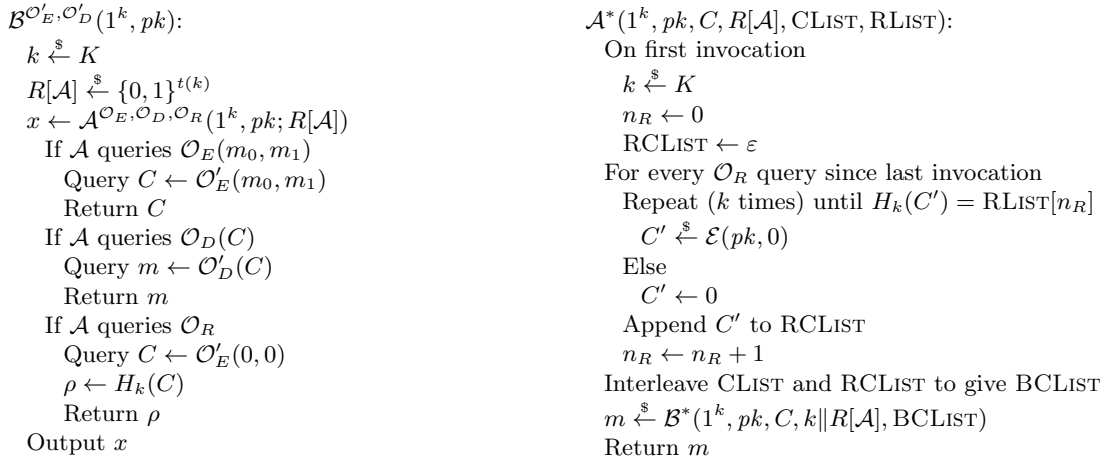


Fig. 12. The ciphertext creator \mathcal{B} and the plaintext extractor \mathcal{A}^* . Note that $R[\mathcal{B}] = k \| R[\mathcal{A}]$ as \mathcal{B} generates no other random values itself. The variable n_R in \mathcal{A}^* counts the total number of randomness oracle queries made by \mathcal{A} and the list RCLIST gives a list of ciphertexts that, when hashed, give the bits in RLIST. The interleave operation in \mathcal{A}^* interleaves CLIST and RCLIST according to the ordering of encryption oracle and randomness oracle queries made by \mathcal{A} . This creates a list BCLIST which is suitable for use with \mathcal{B}^* .

<p>G_1: $\mathcal{O}_D(C)$: $m \leftarrow \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$ Return m</p> <p>\mathcal{O}_R: $\rho \xleftarrow{\\$} \{0, 1\}$ Return ρ</p>	<p>$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$: On first invocation $k \xleftarrow{\\$} K$ $n_R \leftarrow 0$ $\text{RCLIST} \leftarrow \varepsilon$ For every \mathcal{O}_R query since last invocation Repeat (k times) until $H_k(C') = \text{RLIST}[n_R]$ $C' \xleftarrow{\\$} \mathcal{E}(pk, 0)$ Else $C' \leftarrow 0$ Append C' to RCLIST $n_R \leftarrow n_R + 1$ Interleave CLIST and RCLIST to give BCLIST $m \xleftarrow{\\$} \mathcal{B}^*(1^k, pk, C, k \ R[\mathcal{A}], \text{BCLIST})$ Return m</p>
<p>G_2: $\mathcal{O}_D(C)$: $m \leftarrow \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$ Return m</p> <p>$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$: $m \xleftarrow{\\$} \mathcal{B}^*(1^k, pk, C, k \ R[\mathcal{A}], \text{CLIST})$ Return m</p>	<p>\mathcal{O}_R: On first invocation $k \xleftarrow{\\$} K$ $\rho \xleftarrow{\\$} \{0, 1\}$ Repeat (k times) until $H_k(C) = \rho$ $C \xleftarrow{\\$} \mathcal{E}(pk, 0)$ Or $C \leftarrow 0$ Append C to CLIST Return ρ</p>
<p>G_3: $\mathcal{O}_D(C)$: $m \leftarrow \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$ Return m</p> <p>$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$: $m \xleftarrow{\\$} \mathcal{B}^*(1^k, pk, C, k \ R[\mathcal{A}], \text{CLIST})$ Return m</p>	<p>\mathcal{O}_R: On first invocation $k \xleftarrow{\\$} K$ $\rho \xleftarrow{\\$} \{0, 1\}$ Repeat until $H_k(C) = \rho$ $C \xleftarrow{\\$} \mathcal{E}(pk, 0)$ Append C to CLIST Return ρ</p>
<p>G_4: $\mathcal{O}_D(C)$: $m \leftarrow \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$ Return m</p> <p>$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}, \text{RLIST})$: $m \xleftarrow{\\$} \mathcal{B}^*(1^k, pk, C, k \ R[\mathcal{A}], \text{CLIST})$ Return m</p>	<p>\mathcal{O}_R: On first invocation $k \xleftarrow{\\$} K$ $C \xleftarrow{\\$} \mathcal{E}(pk, 0)$ $\rho \leftarrow H_k(C)$ Append C to CLIST Return ρ</p>

Fig. 13. Definition of games $G_1 - G_4$ for the proof of Theorem 3.14

$G_5: \mathcal{O}_D(C):$ <div style="border: 1px solid black; padding: 2px; display: inline-block; margin: 2px;"> $m \leftarrow \mathcal{D}(sk, C)$ </div> Return m	$\mathcal{O}_R:$ On first invocation $k \xleftarrow{\$} K$ $C \xleftarrow{\$} \mathcal{E}(pk, 0)$ $\rho \leftarrow H_k(C)$ Append C to CLIST Return ρ
$G_6: \mathcal{O}_D(C):$ $m \leftarrow \mathcal{D}(sk, C)$ Return m	$\mathcal{O}_R:$ On first invocation $k \xleftarrow{\$} K$ $C \xleftarrow{\$} \mathcal{E}(pk, 0)$ $\rho \leftarrow H_k(C)$ Return ρ
$G_7: \mathcal{O}_D(C):$ $m \leftarrow \mathcal{D}(sk, C)$ Return m	$\mathcal{O}_R:$ <div style="border: 1px solid black; padding: 2px; display: inline-block; margin: 2px;"> $\rho \xleftarrow{\\$} \{0, 1\}$ </div> Return ρ

Fig. 14. Definition of games $G_5 - G_7$ for the proof of Theorem 3.14

We now prove that \mathcal{A}^* simulates the decryption oracle for \mathcal{A} . Fix a PPT distinguisher algorithm D . We will define a series of games G_i in which \mathcal{A} outputs a variable x and let W_i be the event that $D(x) = 1$ in game G_i . The games are summarised in Figures 13 and 14.

Game G_1 : G_1 is set to be the $\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, D}^{\text{FAKE-2PA2}}(k)$ game.

Game G_2 : G_2 is similar to G_1 except that some of the components of the plaintext extractor \mathcal{A}^* into the randomness extractor \mathcal{O}_R . G_1 and G_2 are identical except for one (subtle) exception. The ciphertext creator \mathcal{A} is forbidden from submitting a ciphertext $C \in \text{CLIST}$ to the decryption oracle. This means that, in G_2 , the ciphertext creator is forbidden from querying the decryption oracle on $C \in \text{RCLIST}$. The two games are identical if this does not occur. Throughout this proof, z will be the distribution of $\mathcal{E}(pk, 0)$ on \mathcal{C} . The elements of RCLIST are distributed according to z and are unknown to \mathcal{A} . By Lemma 3.12, the probability that some specific decryption oracle query C is equal to some specific ciphertext $C' \in \text{RCLIST}$ is bounded by $\sqrt{\kappa(z)}$ and $\kappa(z)$ is bounded by $\text{Adv}_{\mathcal{B}'}^{\text{IND}}(k)$ for some IND-CPA adversary \mathcal{B}' . Therefore, $|\Pr[W_1] - \Pr[W_2]| \leq q_D q_R \sqrt{\text{Adv}_{\mathcal{B}'}^{\text{IND}}(k)}$.

Game G_3 : G_3 is similar to G_2 except that \mathcal{O}_R continues to generate ciphertexts C until it generates one for which $H_k(C) = \rho$. Obviously, G_2 and G_3 are identical as long as \mathcal{O}_R doesn't "abort" and set $C \leftarrow 0$ in G_2 . By the Leftover Hash Lemma (Lemma 3.13), if $C' \xleftarrow{\$} \mathcal{E}(pk, 0)$ and $\rho \xleftarrow{\$} \{0, 1\}$, then $\Delta[(k, H_k(C')), (k, \rho)] \leq \sqrt{\kappa(z)}/2$. Since $\kappa(z) = \text{Adv}_{\mathcal{B}'}^{\text{IND}}(k)$ and Π is IND-CPA secure, we have that $\Pr[H_k(C') = 0], \Pr[H_k(C') = 1] \leq 2/3$ for large enough k . Therefore, for large enough values of k , we have that the probability that we "abort" in G_2 is bounded by $q_R(2/3)^k$ and so $|\Pr[W_2] - \Pr[W_3]| \leq q_R(2/3)^k$.

Game G_4 : G_4 is similar to G_3 except that we again alter the randomness oracle. In G_4 we compute $\rho \leftarrow H_k(C')$ for $C' \xleftarrow{\$} \mathcal{E}(pk, 0)$ rather than $\rho \xleftarrow{\$} \{0, 1\}$. Suppose that \mathcal{O}_R outputs random bits $\rho_1, \dots, \rho_{q_R}$ in G_3 and $\rho'_1, \dots, \rho'_{q_R}$ in G_4 . By the Leftover Hash Lemma, we have that

$$\Delta[(k, \rho_1, \dots, \rho_{q_R}), (k, \rho'_1, \dots, \rho'_{q_R})] \leq q_R \sqrt{\kappa(z)/2} = q_R \sqrt{\text{Adv}_{\mathcal{B}'}^{\text{IND}}(k)/2}.$$

This is a distributional step in game hopping, see Section 2.1, and so $|\Pr[W_4] - \Pr[W_3]| \leq q_R \sqrt{\text{Adv}_{\mathcal{B}'}^{\text{IND}}(k)/2}$.

Game G_5 : A close examination of the inputs to \mathcal{B} in G_4 shows that the event W_4 is the same as the event $\text{EXPT}_{\mathcal{B}, \mathcal{B}^*, \mathcal{P}, D}^{\text{FAKE-2PA2}}(k) = 1$. G_5 is the game in which decryption oracle queries are answered by the real decryption algorithm. So W_5 is the same as $\text{EXPT}_{\mathcal{B}, \mathcal{P}, D}^{\text{REAL-2PA2}}(k) = 1$. Thus, $|\Pr[W_5] - \Pr[W_4]|$ is negligible as Π is 2PA2 plaintext aware.

Game G_6 : G_6 and G_7 reverse the previous changes to the randomness oracle. In G_6 the randomness oracle is changed so that it no longer adds ciphertexts to CLIST. Since decryption oracle queries are answered by $\mathcal{D}(sk, \cdot)$, rather than the plaintext extractor \mathcal{A}^* , the only affect of this change is to allow the ciphertext creator \mathcal{A} to query the decryption oracle on ciphertexts in (the list previously described as) RCLIST. By a similar argument to G_2 , we have that $|\Pr[W_6] - \Pr[W_5]| \leq q_D q_R \sqrt{Adv_{\mathcal{B}'}^{\text{IND}}(k)}$.

Game G_7 : G_7 changes the randomness oracle so that it simply returns a random bit. By a similar argument to G_4 , we have that $|\Pr[W_7] - \Pr[W_6]| \leq q_R \sqrt{Adv_{\mathcal{B}'}^{\text{IND}}(k)}/2$. Now, an examination of G_7 shows that it is identical to the $\text{EXPT}_{\mathcal{A}, \mathcal{P}, \mathcal{D}}^{\text{REAL-2PA2}^+}(k)$ game. Therefore, we have that

$$|\Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, \mathcal{D}}^{\text{FAKE-2PA2}^+}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{P}, \mathcal{D}}^{\text{REAL-2PA2}^+}(k) = 1]| = |\Pr[W_1] - \Pr[W_7]|$$

is negligible and so Π is 2PA2+ is plaintext aware. \square

Corollary 3.15. *If a public-key encryption scheme Π is PA2 plaintext aware, IND-CPA secure and has that $|\mathcal{M}|$ grows faster than any polynomial, then Π is PA2+ plaintext aware (and therefore also PA1+ plaintext aware).*

Proof Since Π is PA2 plaintext aware, it is 2PA2 plaintext aware (by inclusion). Since Π is IND-CPA secure and 2PA2 plaintext aware, it is 2PA2+ plaintext aware (by Theorem 3.14). We also have that, since Π is PA2 plaintext aware, IND-CPA secure and has that $|\mathcal{M}|$ grows faster than any polynomial, Π is LH-IND-CPA (by Corollary 3.5). Lastly, since Π is LH-IND-CPA and 2PA2+, Π is PA2+ (by an argument analogous to Corollary 3.15). \square

3.4 A Strategy for Proving PA2 Plaintext Awareness

In this section, we will prove that a public-key encryption scheme which is PA1+ plaintext aware and simulatable is PA2 plaintext aware. This is logical: if the encryption scheme is simulatable then ciphertexts are indistinguishable from random bitstrings. Hence, access to an encryption oracle is indistinguishable from access to a randomness oracle. This gives the only known strategy for proving that schemes are PA2 plaintext aware: one proves that the scheme is both simulatable and that it is PA1+ plaintext aware.

Theorem 3.16. *If Π is simulatable and PA1+ plaintext aware, then it is PA2 plaintext aware.*

Proof Suppose that $\Pi = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ is simulatable with simulator (f, f^{-1}, ℓ) and let \mathcal{A} be a PA2 ciphertext creator. We define a PA1+ ciphertext creator \mathcal{B} as in Figure 15. Since Π is PA1+ plaintext aware, there exists a PA1+ plaintext extractor \mathcal{B}^* for \mathcal{B} . We use \mathcal{B}^* to construct a PA2 plaintext extractor \mathcal{A}^* for \mathcal{A} as in Figure 15.

This proof is similar to, but slightly simpler than, Theorem 3.14. Fix a PPT distinguisher D and let W_i be the event that $D(x) = 1$ in game G_i . The games are shown in Figure 16.

Game G_1 : G_1 is the $\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, \mathcal{P}, \mathcal{D}}^{\text{FAKE-PA2}}(k)$ game.

Game G_2 : G_2 is similar to G_1 except that the encryption oracle \mathcal{O}_E returns $f(r)$ for $r \xleftarrow{\$} \{0, 1\}^\ell$ rather than $\mathcal{E}(pk, \mathcal{P}(\alpha))$. Any difference between $\Pr[W_1]$ and $\Pr[W_2]$ gives rise to an attacker \mathcal{B}' against the computational ciphertext-space simulatable property of the encryption scheme. The attacker \mathcal{B}' is defined as follows:

$$\begin{aligned} & \mathcal{B}'^{\mathcal{O}'_E, \mathcal{D}(sk, \cdot)}(1^k, pk): \\ & x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{O}_D}(1^k, pk) \\ & \text{If } \mathcal{A} \text{ queries } \mathcal{O}_E \text{ on } \alpha \\ & \quad m \leftarrow \mathcal{P}(\alpha) \\ & \quad \text{Query } C \xleftarrow{\$} \mathcal{O}'_E(m) \\ & \quad \text{Return } C \\ & \text{If } \mathcal{A} \text{ queries } \mathcal{O}_D \text{ on } C \\ & \quad m \xleftarrow{\$} \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST}) \\ & \quad \text{Return } m \\ & \text{Output } D(x) \end{aligned}$$

$\mathcal{B}^{\mathcal{O}'_D, \mathcal{O}'_R}(1^k, pk):$ $x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{O}_D}(1^k, pk)$ If \mathcal{A} queries \mathcal{O}_D on C Query $m \leftarrow \mathcal{O}'_D(C)$ Return m If \mathcal{A} queries \mathcal{O}_E on α Query $r \leftarrow \mathcal{O}'_R$ $C \leftarrow f(r)$ Return C Output x	$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ On first invocation: $\text{RLIST} \leftarrow \varepsilon$ $n_C \leftarrow 0$ Repeat until $n_C = \text{CLIST} $ $r' \xleftarrow{\$} f^{-1}(\text{CLIST}[n_C])$ Append r' to RLIST $n_C \leftarrow n_C + 1$ $m \xleftarrow{\$} \mathcal{B}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$ Return m
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Fig. 15. The ciphertext creator \mathcal{B} and plaintext extractor \mathcal{A}^* . Without loss of generality, we assume that the randomness oracle \mathcal{O}'_R returns blocks of $\ell(k)$ random bits. This could be achieved by querying a one-bit oracle $\ell(k)$ times. The variable n_C counts the total number of ciphertexts on CLIST and RLIST contains a list of random bits which could have given rise to CLIST under the action of f .

The event W_1 is equivalent to $\text{EXPT}_{\mathcal{B}'}^{\text{PKE-1}}(k) = 1$ and the event W_2 is equivalent to $\text{EXPT}_{\mathcal{B}'}^{\text{PKE-0}}(k) = 1$. Hence, $|\Pr[W_1] - \Pr[W_2]| \leq \text{Adv}_{\mathcal{B}'}^{\text{PKE}}(k)$ is negligible. (Note that \mathcal{B}' does not make use of its decryption oracle in this step. This will not be the case when we make an analogous game hop in G_5 .)

Game G_3 : G_3 is similar to G_2 except that the randomness oracle constructs RLIST rather than the plaintext extractor \mathcal{A}^* . In G_2 , the element $f^{-1}(f(r))$ is added to RLIST . In G_3 , the element r is added to RLIST . Any difference between $\Pr[W_2]$ and $\Pr[W_3]$ gives rise to an attacker \mathcal{B}^* against the computational random-set simulatable property of the encryption scheme. The attacker \mathcal{B}^* is defined as follows:

$$\mathcal{B}^{*\mathcal{O}'_R, \mathcal{D}(sk, \cdot)}(1^k, pk):$$

$$x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_E, \mathcal{O}_D}(1^k, pk)$$

If \mathcal{A} queries \mathcal{O}_E on α
 Query $r \leftarrow \mathcal{O}'_R$
 $C \leftarrow f(r)$
 Append r to RLIST
 Return C
 If \mathcal{A} queries \mathcal{O}_D on C
 $m \xleftarrow{\$} \mathcal{B}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$
 Return m
 Output $D(x)$

The randomness oracle \mathcal{O}'_R returns either $r \xleftarrow{\$} \{0, 1\}^\ell$ or $f^{-1}(f(r))$ for $r \xleftarrow{\$} \{0, 1\}^\ell$ depending on the random-set simulatable game. In both cases, $f(r)$ is returned to the ciphertext creator \mathcal{A} , as $f(f^{-1}(f(r))) = f(r)$ since $f(f^{-1}(C)) = C$ for all $C \in \mathcal{C}$. The event W_2 is identical to $\text{EXPT}_{\mathcal{B}^*}^{\text{RAND-0}}(k) = 1$ and the event W_3 is identical to $\text{EXPT}_{\mathcal{B}^*}^{\text{RAND-1}}(k) = 1$. Thus, $|\Pr[W_3] - \Pr[W_2]| \leq \text{Adv}_{\mathcal{B}^*}^{\text{RAND}}(k)$ is negligible by the random-set simulatable property.

Game G_4 : An examination of G_3 shows that it is identical to $\text{EXPT}_{\mathcal{B}, \mathcal{B}^*, D}^{\text{FAKE-PA1+}}(k)$. G_4 changes the decryption oracle so that it answers queries using the real decryption algorithm, i.e. G_4 is $\text{EXPT}_{\mathcal{B}, D}^{\text{REAL-PA1+}}(k)$. Since Π is PA1+ , we have that $|\Pr[W_4] - \Pr[W_3]| \leq \text{Adv}_{\mathcal{A}, \mathcal{A}^*, D}^{\text{PA1+}}(k)$ is negligible.

Game G_5 : G_5 changes the action of the encryption oracle \mathcal{O}_E back to its original state, i.e. the encryption oracle returns $\mathcal{E}(pk, \mathcal{P}(\alpha))$. A similar argument to that in G_2 shows that any difference between $\Pr[W_4]$ and $\Pr[W_5]$ leads to an attacker \mathcal{B}^\dagger against the computational ciphertext-space simulatable property of the encryption scheme (although in this case decryption oracle queries made by \mathcal{A} are handled using \mathcal{B}^\dagger 's decryption oracle). Thus, $|\Pr[W_5] - \Pr[W_4]| \leq \text{Adv}_{\mathcal{B}^\dagger}^{\text{PKE}}(k)$ is negligible.

$G_1: \mathcal{O}_D(C):$ $m \stackrel{\$}{\leftarrow} \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ Return m $\mathcal{O}_E(\alpha):$ $m \stackrel{\$}{\leftarrow} \mathcal{P}(\alpha)$ $C \stackrel{\$}{\leftarrow} \mathcal{E}(pk, C)$ Return C	$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ On first invocation: $\text{RLIST} \leftarrow \varepsilon$ $n_C \leftarrow 0$ Repeat until $n_C = \text{CLIST} $ $r' \stackrel{\$}{\leftarrow} f^{-1}(\text{CLIST}[n_C])$ Append r' to RLIST $n_C \leftarrow n_C + 1$ $m \stackrel{\$}{\leftarrow} \mathcal{B}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$ Return m
$G_2: \mathcal{O}_D(C):$ $m \stackrel{\$}{\leftarrow} \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ Return m $\mathcal{O}_E(\alpha):$ <div style="border: 1px solid black; padding: 2px; display: inline-block;">$r \stackrel{\\$}{\leftarrow} \{0, 1\}^{\ell(k)}$</div> <div style="border: 1px solid black; padding: 2px; display: inline-block;">$C \leftarrow f(r)$</div> Return C	$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ On first invocation: $\text{RLIST} \leftarrow \varepsilon$ $n_C \leftarrow 0$ Repeat until $n_C = \text{CLIST} $ $r' \stackrel{\$}{\leftarrow} f^{-1}(\text{CLIST}[n_C])$ Append r' to RLIST $n_C \leftarrow n_C + 1$ $m \stackrel{\$}{\leftarrow} \mathcal{B}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$ Return m
$G_3: \mathcal{O}_D(C):$ $m \stackrel{\$}{\leftarrow} \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ Return m $\mathcal{O}_E(\alpha):$ $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell(k)}$ $C \leftarrow f(r)$ <div style="border: 1px solid black; padding: 2px; display: inline-block;">Append r to RLIST</div> Return C	$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{CLIST})$ $m \stackrel{\$}{\leftarrow} \mathcal{B}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$ Return m
$G_4: \mathcal{O}_D(C):$ <div style="border: 1px solid black; padding: 2px; display: inline-block;">$m \leftarrow \mathcal{D}(sk, C)$</div> Return m	$\mathcal{O}_E(\alpha):$ $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell(k)}$ $C \leftarrow f(r)$ Return C
$G_5: \mathcal{O}_D(C):$ $m \leftarrow \mathcal{D}(sk, C)$ Return m	$\mathcal{O}_E(\alpha):$ <div style="border: 1px solid black; padding: 2px; display: inline-block;">$m \stackrel{\\$}{\leftarrow} \mathcal{P}(\alpha)$</div> <div style="border: 1px solid black; padding: 2px; display: inline-block;">$C \stackrel{\\$}{\leftarrow} \mathcal{E}(pk, m)$</div> Return C

Fig. 16. Definition of games G_1 and G_5 for the proof of Theorem 3.16

However, G_5 is identical to $\text{EXPT}_{\mathcal{A},\mathcal{P},D}^{\text{REAL-PA2}}(k)$. So

$$|\Pr[\text{EXPT}_{\mathcal{A},\mathcal{A}^*,\mathcal{P},D}^{\text{FAKE-PA2}}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A},\mathcal{P},D}^{\text{REAL-PA2}}(k) = 1]| = |\Pr[W_1] - \Pr[W_5]|$$

is negligible and we conclude that Π is PA2 plaintext aware. \square

A similar argument (with more complex book-keeping) can be used to prove that a public-key encryption scheme which is simulatable and PA1+ plaintext aware is necessarily PA2+ plaintext aware. However, we will be content with the basic result.

The next part of our strategy is specific to hybrid public-key encryption schemes. We begin by defining a notion of PA1+ plaintext awareness for KEMs via the two games given in Figure 17 (which make use of a ciphertext creator \mathcal{A} , a plaintext extractor \mathcal{A}^* and a distinguisher algorithm D). The advantage is defined as

$$\text{Adv}_{\mathcal{A},\mathcal{A}^*,D}^{\text{KEM-PA1}^+}(k) = |\Pr[\text{EXPT}_{\mathcal{A},D}^{\text{REAL-KEM-PA1}^+}(k) = 1] - \Pr[\text{EXPT}_{\mathcal{A},\mathcal{A}^*,D}^{\text{FAKE-KEM-PA1}^+}(k) = 1]|.$$

$\text{EXPT}_{\mathcal{A},D}^{\text{REAL-KEM-PA1}^+}(k)$: $(pk, sk) \xleftarrow{\$} \text{Gen}(1^k)$ $x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_D, \mathcal{O}_R}(1^k, pk)$ If \mathcal{A} queries \mathcal{O}_R $\rho \xleftarrow{\$} \{0, 1\}$ Return ρ If \mathcal{A} queries $\mathcal{O}_D(C)$ $K \leftarrow \text{Decap}(sk, C)$ Return K $b \xleftarrow{\$} D(x)$ Output b	$\text{EXPT}_{\mathcal{A},\mathcal{A}^*,D}^{\text{FAKE-KEM-PA1}^+}(k)$: $(pk, sk) \xleftarrow{\$} \text{Gen}(1^k)$ $x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}_D, \mathcal{O}_R}(1^k, pk)$ If \mathcal{A} queries \mathcal{O}_R $\rho \xleftarrow{\$} \{0, 1\}$ Append ρ to RLIST Return ρ If \mathcal{A} queries $\mathcal{O}_D(C)$ $K \xleftarrow{\$} \mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$ Return K $b \xleftarrow{\$} D(x)$ Output b
--	---

Fig. 17. The PA1+ plaintext awareness security games for KEMs

Definition 3.17. A KEM Π_K is PA1+ plaintext aware if for every PPT ciphertext creator \mathcal{A} there exists a PPT plaintext extractor \mathcal{A}^* such that for all PPT distinguisher algorithms D we have that $\text{Adv}_{\mathcal{A},\mathcal{A}^*,D}^{\text{KEM-PA1}^+}(k)$ is negligible.

We now simplify the process of proving PA1+ plaintext awareness in the case of hybrid encryption schemes by showing that a hybrid encryption scheme Π is PA1+ plaintext aware if and only if its KEM Π_K is PA1+ plaintext aware. This proof technique does not immediately extend to PA2 plaintext awareness and this issue is further discussed by Jiang and Wang [18].

Lemma 3.18. Suppose Π is a public-key encryption scheme composed of a KEM scheme Π_K and a DEM scheme Π_D . Π is PA1+ plaintext aware if and only if Π_K is PA1+ plaintext aware.

Proof Let \mathcal{A} be a PA1+ ciphertext creator for Π and suppose that the DEM Π_D is a (deterministic) DEM of the form (ENC, DEC). We may define a PA1+ ciphertext creator \mathcal{B} for Π_K as in Figure 18. Since Π_K is PA1+, there exists a plaintext extractor \mathcal{B}^* for \mathcal{B} . We define a plaintext extractor \mathcal{A}^* for Π in Figure 18. An examination of the games easily shows that for any distinguisher algorithm D , we have $\text{EXPT}_{\mathcal{A},D}^{\text{REAL-PA1}^+}(k) = \text{EXPT}_{\mathcal{B},D}^{\text{REAL-KEM-PA1}^+}(k)$ and $\text{EXPT}_{\mathcal{A},\mathcal{A}^*,D}^{\text{FAKE-PA1}^+}(k) = \text{EXPT}_{\mathcal{B},\mathcal{B}^*,D}^{\text{FAKE-KEM-PA1}^+}(k)$, which means that Π is PA1+ if and only if Π_K is PA1+. \square

This now gives a complete strategy to prove that a hybrid encryption scheme, consisting of a KEM Π_K and a DEM Π_D , is PA2 plaintext aware:

- Prove that Π_K is PA1+ plaintext aware.

$\mathcal{B}^{\mathcal{O}'_D, \mathcal{O}'_R}(1^k, pk)$ $x \xleftarrow{\$} \mathcal{A}^{\mathcal{O}'_D, \mathcal{O}'_R}$ If \mathcal{A} queries \mathcal{O}'_R Query $r \leftarrow \mathcal{O}'_R$ Return r If \mathcal{A} queries $\mathcal{O}'_D(C)$ Parse C as (C_1, C_2) Query $K \leftarrow \mathcal{O}'_D(C_1)$ $m \leftarrow \text{DEC}_K(C_2)$ Return m Output x	$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$ Parse C as (C_1, C_2) $K \xleftarrow{\$} \mathcal{B}^*(1^k, pk, C_1, R[\mathcal{A}], \text{RLIST})$ $m \leftarrow \text{DEC}_K(C_2)$ Output m
--	---

Fig. 18. The ciphertext creator \mathcal{B} and plaintext extractor \mathcal{A}^*

- Prove that both Π_K and Π_D are simulatable.

Recall that several DEMs have already been shown to be simulatable, most notable the encrypt-then-MAC DEM where encryption is performed using an ideal cipher in counter mode and the MAC is performed using an ideal cipher in CBC-MAC mode. Hence, we may concentrate on proving that Π_K is PA1+ and simulatable.

4 The Cramer-Shoup Encryption Scheme

In this section we will use our strategy to show the hybrid Cramer-Shoup encryption schemes [10] is PA2 (under a variety of assumptions including the interactive Diffie-Hellman Knowledge (DHK) assumption⁵).

4.1 Assumptions

The Cramer-Shoup encryption scheme is defined over a group \mathbb{G} with a generator g of prime order q . We will require that the group is statistically simulatable (as a set), that the DDH problem is hard on \mathbb{G} and that the DHK assumption holds on \mathbb{G} .

Definition 4.1 (DDH). *The DDH advantage of an attacker \mathcal{A} is defined as*

$$\text{Adv}_{\mathcal{A}}^{\text{DDH}}(k) = \left| \Pr \left[b' = 1 : \begin{array}{l} x, y \xleftarrow{\$} \mathbb{Z}_q, \\ b' \xleftarrow{\$} \mathcal{A}(g, g^x, g^y, g^{xy}) \end{array} \right] - \Pr \left[b' = 1 : \begin{array}{l} x, y, z \xleftarrow{\$} \mathbb{Z}_q, \\ b' \xleftarrow{\$} \mathcal{A}(g, g^x, g^y, g^z) \end{array} \right] \right|.$$

The DDH problem is hard if every PPT attacker \mathcal{A} has negligible advantage.

The DHK assumption was first introduced by D amgaard [11]. It is a very strong, interactive assumption which is designed to capture the intuition that it does not seem to be possible to create a Diffie-Hellman tuple (g, a, g^x, a^x) from (g, a) without knowing x . Let $DH = \{(g, g^x, g^y, g^{xy}) : x, y \in \mathbb{Z}_q\}$. We define DHK security via the security game shown in Figure 19.⁶

⁵ This is also sometimes known as the Knowledge-of-Exponent assumption (KEA).

⁶ Like many cryptographic assumptions, we are unlikely to achieve advantage zero, i.e. it is unlikely that we can find a perfect extractor. In a statistically simulatable group, the existence of a perfect extractor implies that the DDH problem is easy. If (Gf, Gf^{-1}, ℓ) is a simulator for \mathbb{G} , then we may construct a DHK attacker \mathcal{A} which generates $r, r' \xleftarrow{\$} \{0, 1\}^\ell$, computes $b \leftarrow Gf(r)$ and $c \leftarrow Gf(r')$, and submits (b, c) to the extractor. If there exists a perfect DHK extractor \mathcal{A}^* for \mathcal{A} then \mathcal{A}^* always correctly returns x such that $b = g^x$ whenever $(g, a, b, c) \in DH$. We may use this to build a DDH solver $\mathcal{B}(g, a, b, c)$ which computes $R = Gf^{-1}(b) \| Gf^{-1}(c)$, runs $x \xleftarrow{\$} \mathcal{A}^*(g, a, b, c, R)$, and returns 1 if $b = g^x$ and $c = a^x$.

$\text{EXPT}_{\mathcal{A}, \mathcal{A}^*}^{\text{DHK}}(k)$:
 $a \xleftarrow{\$} \mathbb{G}$
 $\mathcal{A}^{\mathcal{O}}(1^k, g, a)$
 If \mathcal{A} queries $\mathcal{O}(b, c)$
 $x \xleftarrow{\$} \mathcal{A}^*(g, a, b, c, R[\mathcal{A}])$
 If $(g, a, b, c) \in \text{DH}$ and $b \neq g^x$ then output 1
 Else return x
 Output 0

Fig. 19. The security game for the Diffie-Hellman Knowledge (DHK) assumption. \mathcal{A}^* is run as a stateful algorithm (which retains state between invocations).

Definition 4.2 (DHK). *The DHK advantage of an attacker \mathcal{A} and an extractor \mathcal{A}^* is defined as $\text{Adv}_{\mathcal{A}, \mathcal{A}^*}^{\text{DHK}}(k) = \Pr[\text{EXPT}_{\mathcal{A}, \mathcal{A}^*}^{\text{DHK}}(k) = 1]$. The DHK assumption holds if for every PPT attacker \mathcal{A} there exists a PPT extractor algorithm \mathcal{A}^* such that $\text{Adv}_{\mathcal{A}, \mathcal{A}^*}^{\text{DHK}}(k)$ is negligible.*

So, in order to prove that the Cramer-Shoup encryption scheme is plaintext aware, we require candidate groups which are simulatable and on which the DDH and DHK assumptions are believed to hold. As a tool, we begin by showing that families of sets $S = ((S_i)_{i \in I_k})_{k \in \mathbb{N}}$ of the form $S_i = \mathbb{Z}_{N_i}$ for all $i \in I_k$, where $2^{k-1} < N_i < 2^k$, are statistically simulatable. We define a simulator $\Pi_S = (f, f^{-1}, \ell)$ where $\ell = 2k$ and

$f(i, r)$: $x \leftarrow r \bmod N_i$ Output x	$f^{-1}(i, s)$: $\alpha_i \leftarrow \lfloor 2^{2k}/N_i \rfloor$ $t \xleftarrow{\$} \{0, \dots, \alpha_i - 1\}$ $r \leftarrow t\alpha_i + s$ Output r
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Lemma 4.3. *The simulator Π_S witnesses the fact that S is simulatable, i.e. Π_S is statistically random-set and statistically image-set simulatable.*

Proof Let $\alpha_i = \lfloor 2^{2k}/N_i \rfloor$ and $\beta_i = 2^{2k} \bmod N_i$ (i.e. $2^{2k} = \alpha_i N_i + \beta_i$). Let U be the uniform distribution on S_i and let Y be the distribution given by $f(i, r)$ for $r \xleftarrow{\$} \{0, 1\}^\ell$. Split S_i into two disjoint sets S_i^+ and S_i^- where $S_i^+ = \{0, \dots, \beta_i - 1\}$ and $S_i^- = \{\beta_i, \dots, N_i - 1\}$. Hence,

$$|\Pr[Y = s] - \Pr[U = s]| = \begin{cases} \left| \frac{\alpha_i + 1}{2^{2k}} - \frac{1}{N_i} \right| & \text{if } s \in S_i^+ \\ \left| \frac{\alpha_i}{2^{2k}} - \frac{1}{N_i} \right| & \text{if } s \in S_i^- \end{cases}$$

Since $2^{2k} - \alpha_i N_i = \beta_i$, we have that $|\Pr[Y = s] - \Pr[U = s]| \leq 1/2^{2k}$ for all $s \in S_i$. Thus, $\Delta[U, Y] \leq 1/2^k$ and S is statistically image-set simulatable.

Now let U' be the uniform distribution on $\{0, 1\}^\ell$ and Y' be the distribution of $f^{-1}(i, f(i, r))$ for $r \xleftarrow{\$} \{0, 1\}^\ell$. We have

$$|\Pr[Y' = r'] - \Pr[U' = r']| = \begin{cases} \left| \frac{\alpha_i + 1}{\alpha_i 2^{2k}} - \frac{1}{2^{2k}} \right| & \text{if } r' < \alpha_i N_i \text{ and } r' \bmod N_i \in S_i^+ \\ \left| \frac{\alpha_i}{\alpha_i 2^{2k}} - \frac{1}{2^{2k}} \right| & \text{if } r' < \alpha_i N_i \text{ and } r' \bmod N_i \in S_i^- \\ \frac{1}{2^{2k}} & \text{if } r' \geq \alpha_i N_i \end{cases}$$

There exists $\alpha_i \beta_i \leq \alpha_i N_i$ values for r' with $r' < \alpha_i N_i$ and $r' \bmod N_i \in S_i^+$ and there exists $\beta_i < N_i$ values r' with $r' \geq \alpha_i N_i$. Thus, $\Delta[U', Y'] \leq 1/2^k$ and S is statistically random-set simulatable. \square

This is not sufficient to show that cryptographic groups are statistically simulatable; however, we may use this result as a bridge to results about cryptographic groups.

Lemma 4.4. *Suppose $S = ((S_i)_{i \in I_k})_{k \in \mathbb{N}}$ is statistically simulatable family of finite sets and that $S' = ((S'_i)_{i \in I_k})_{k \in \mathbb{N}}$ satisfies $S'_i \subseteq S_i$ for all $i \in I_k$ and $k \in \mathbb{N}$. Let $\mu(k, i) = |S'_i|/|S_i|$ and $\nu(k, i) = 1 - \mu(k, i)$, and suppose that there exists a polynomial $p(k)$ such that $\mu(k, i), \nu(k, i) \geq 1/p(k)$ for all sufficiently large k and $i \in I_k$. Suppose further that there exists polynomial-time algorithm $U(1^k, i)$ which outputs $\mu(k, i)$ and that membership of S_i and S'_i are polynomial-time decidable. Then S' is a statistically simulatable family of sets.*

Proof Suppose (f, f^{-1}, ℓ) is a statistical simulator for S and let $Ber(\mu)$ be a Bernoulli distribution which outputs 1 with probability μ . We define a new simulator (f', f'^{-1}, ℓ') where $\ell'(k) = kp(k)\ell(k)$ and the remaining algorithms are given in Figure 20.

$f'(i, r)$: Parse r as $r_1 \ \dots \ r_{\ell'}$ where $r_j \in \{0, 1\}^{\ell'}$ For $j = 1, 2, \dots, kp(k)$: If $f(i, r_j) \in S'_i$ Output $f(i, r_j)$ Output 0	$f'^{-1}(i, s)$: $r \leftarrow \varepsilon$ $\mu \leftarrow U(1^k, i)$ $found \leftarrow \mathbf{false}$ For $j = 1, 2, \dots, kp(k)$: $\rho \stackrel{\$}{\leftarrow} Ber(\mu)$ If $\rho = 1$ and $found = \mathbf{false}$ $r_j \leftarrow f^{-1}(i, s)$ Append r_j to r $found \leftarrow \mathbf{true}$	If $\rho = 0$ and $found = \mathbf{false}$ Repeat $(kp(k)$ times) until $s'_j \notin S_i$ $r_j \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell}$ $s'_j \leftarrow f(i, r_j)$ Else $s'_j \leftarrow f(i, 0^{\ell})$ Append $f^{-1}(i, s'_j)$ to r If $found = \mathbf{true}$ $r_j \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell(k)}$ Append r_j to r Output r
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Fig. 20. Simulator f' and f'^{-1} for sets $S' \subseteq S$

This proof can thought of as “distributional game hopping” and makes liberal use of the results of Section 2.1. We define a number of distributions – prime distributions (e.g. X') are associated with (f', f'^{-1}, ℓ') and un-prime distributions are associated with (f, f^{-1}, ℓ) . For example,

- $U_\ell, U_{\ell'}, U_T$ refer to the uniform distributions over the sets $\{0, 1\}^\ell, \{0, 1\}^{\ell'}, T$ respectively. Note that T may be any set including S_i, S'_i and $S_i \setminus S'_i$.
- X refers to the distribution $f(i, r)$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^\ell$.
- Y refers to the distribution $f^{-1}(i, f(i, r))$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^\ell$.
- Z refers to the distribution $f^{-1}(i, s)$ for $s \stackrel{\$}{\leftarrow} S_i$.
- X' refers to the distribution $f'(i, r)$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell'}$ and \bar{X}' refers to the distribution $f'(i, r)$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell'}$ conditioned on the fact that f' does not “abort” and output 0. Note that $\Delta[X', \bar{X}'] \leq 1/2^k$.
- Y' refers to the distribution $f'^{-1}(i, f(i, r))$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell'}$ and \bar{Y}' refers to the distribution $f'^{-1}(i, f(i, r))$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell'}$ conditioned on the fact that f' does not “abort” and output 0. Note that $\Delta[Y', \bar{Y}'] \leq 1/2^k$.
- Z' refers to the distribution $f'^{-1}(i, s)$ for $s \stackrel{\$}{\leftarrow} S'_i$ and \bar{Z}' refers to the distribution $f'^{-1}(i, s)$ for $s \stackrel{\$}{\leftarrow} S'_i$ conditioned on the fact that f'^{-1} does not “abort” and output 0. Note that $\Delta[Z', \bar{Z}'] \leq 1/2^k$.

In order to show that S' is statistically image-set simulatable, we need to show that $\Delta[U_{S'_i}, X']$ is negligible, but this follows simply as

$$\Delta[U_{S'_i}, X'] \leq \Delta[U_{S'_i}, \bar{X}'] + \Delta[\bar{X}', X'] \leq \Delta[U_{S'_i}, \bar{X}'] + 1/2^k \leq p(k)\Delta[U_S, X] + 1/2^k.$$

It is more difficult to show that S'_i is statistically random-set simulatable. We make use of the following simple fact:

$$\Delta[U_\ell, Z] \leq \Delta[U_\ell, Y] + \Delta[Y, Z] \leq \Delta[U_\ell, Y] + \Delta[X, U_{S_i}].$$

Thus, $\Delta[U_\ell, Z]$ is negligible (as S is statistically image-set and random-set simulatable). This can be interpreted as saying that the distribution $f(i, s)$ for $s \stackrel{\$}{\leftarrow} S_i$ is statistically close to uniform on $\{0, 1\}^\ell$.

Consider $r' \stackrel{\$}{\leftarrow} f'(i, f(i, r))$ for $r \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell'}$. We may split r' into “blocks” $r'_1 \parallel \dots \parallel r'_{kp(k)}$ where each $r'_j \in \{0, 1\}^\ell$. Let λ be the smallest integer such that $f(i, r'_\lambda) \in S'_i$. We claim that the distribution of each “block” r_j is statistically close to an independent copy of the uniformly random distribution U_ℓ . This is trivially true for blocks with $j > \lambda$.

In order to show that S' is statistically random-set simulatable, we have to show that $\Delta[U_{\ell'}, Y']$ is negligible. We begin by noting that

$$\begin{aligned} \Delta[U_{\ell'}, Y'] &\leq \Delta[U_{\ell'}, \bar{Y}'] + \Delta[\bar{Y}', Y'] && \text{(as } \Delta[\bar{Y}', Y'] \leq 1/2^k) \\ &\leq \Delta[U_{\ell'}, \bar{Y}'] + 1/2^k \\ &\leq \Delta[U_{\ell'}, Z'] + \Delta[Z', \bar{Y}'] + 1/2^k \\ &\leq \Delta[U_{\ell'}, Z'] + \Delta[U_{S'_i}, X'] + 1/2^k && \text{(as } Z' \text{ and } \bar{Y}' \text{ differ only} \\ & && \text{by their inputs)} \\ &\leq \Delta[U_{\ell'}, \bar{Z}'] + \Delta[\bar{Z}', Z'] + \Delta[U_{S'_i}, X'] + 1/2^k \\ &\leq \Delta[U_{\ell'}, \bar{Z}'] + \Delta[U_{S'_i}, X'] + 2/2^k && \text{(as } \Delta[\bar{Z}', Z'] \leq 1/2^k) \end{aligned}$$

We have already shown that $\Delta[U_{S'_i}, X']$ is negligible and so it suffices to bound $\Delta[U_{\ell'}, \bar{Z}']$. In other words, it suffices to show that the distribution $f'^{-1}(i, s)$ for $s \stackrel{\$}{\leftarrow} S'_i$ (conditioned on the fact that f'^{-1} does not abort and output a block $r_j = 0^\ell$) is statistically close to uniform.

We now change the action of the simulator $f'^{-1}(i, s)$ as follows:

$\hat{f}^{-1}(i, s)$: $r \leftarrow \varepsilon$ $\mu \leftarrow U(1^k, i)$ $found \leftarrow \mathbf{false}$ For $j = 1, 2, \dots, \ell'(k)$: $\rho \stackrel{\$}{\leftarrow} \text{Ber}(\mu)$ If $\rho = 1$ and $found = \mathbf{false}$ $r_j \leftarrow f^{-1}(i, s)$ Append r_j to r $found \leftarrow \mathbf{true}$	If $\rho = 0$ and $found = \mathbf{false}$ <div style="border: 1px solid black; display: inline-block; padding: 2px 5px;">$s'_j \stackrel{\\$}{\leftarrow} S_i \setminus S'_i$</div> Append $f^{-1}(i, s'_j)$ to r If $found \leftarrow \mathbf{true}$ $r_j \stackrel{\$}{\leftarrow} \{0, 1\}^{\ell(k)}$ Append r_j to r Output r
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Let \hat{Z}' be the distribution of $\hat{f}^{-1}(i, s)$ for $s \stackrel{\$}{\leftarrow} S'_i$. We have

$$\Delta[U_{\ell'}, \bar{Z}'] \leq \Delta[U_{\ell'}, \hat{Z}'] + \Delta[\hat{Z}', \bar{Z}'].$$

It is not immediately obvious how to bound either of these terms. We bound the term $\Delta[\hat{Z}', \bar{Z}']$ by noting that the only difference between the two distributions is in the distribution of the value s'_j . For \bar{Z}' the variable s'_j is distributed as $f(i, r_j)$ for $r_j \stackrel{\$}{\leftarrow} \{0, 1\}^\ell$ (conditioned on the fact that $f(i, r_j) \notin S'_i$); while for \hat{Z}' the variable s'_j is uniformly distributed over $S \setminus S'_j$. By an argument that is almost identical to the proof that S' is statistically image-set simulatable, we have that these two distributions are statistically indistinguishable. Hence, $\Delta[\hat{Z}', \bar{Z}']$ is negligible.

It therefore only remains to prove that $\Delta[U_{\ell'}, \hat{Z}']$ is negligible. In \hat{Z}' , each “block” r'_j output by $\hat{f}^{-1}(i, s)$ is either uniformly randomly distributed or distributed as $f^{-1}(i, s)$ for $s \stackrel{\$}{\leftarrow} S$ (i.e. distributed according to Z). (This is because Z outputs a random element of S'_i with probability μ and a random element of $S_i \setminus S'_i$ with probability ν .) Therefore we have that

$$\Delta[U_{\ell'}, \hat{Z}'] \leq kp(k)\Delta[U_\ell, Z]$$

which is negligible. Thus, S' is statistically random-set simulatable. □

Corollary 4.5. *The following sets are statistically simulatable:*

- The family of sets $(G_k)_{k \in \mathbb{N}}$ of elements of a prime-order multiplicative group $G_k \subseteq \mathbb{F}_{p_k}$ where p_k is a k -bit integer.
- The family of sets $(G_k)_{k \in \mathbb{N}}$ of elements of a prime-order elliptic curve group $G_k \subseteq E(\mathbb{F}_{p_k})$ where E is an elliptic curve and p_k is a k -bit prime.

Proof Let G_k be the elements of a group $G_k \subseteq \mathbb{F}_{p_k}$ of order q_k . Let $\alpha_k = (p_k - 1)/q_k$ be the index of G_k in $\mathbb{F}_{p_k}^*$. We have shown that $\mathbb{F}_{p_k} = \mathbb{Z}_{p_k}$ is statistically simulatable, and G_k has the property that $\frac{1}{\alpha_k} \leq |G_k|/|\mathbb{F}_{p_k}| \leq 1 - \frac{1}{\alpha_k}$, therefore G_k is simulatable. The proof for elliptic curve groups is similar. \square

A similar result can easily be derived which shows that if S_i is statistically simulatable and $\nu(k, i)$ is negligible (as a function of k) then S'_i is statistically simulatable. This suffices to show that \mathbb{Z}_N^* is statistically simulatable when N is the product of two k -bit primes.

The Cramer-Shoup also requires the use of a target collision resistant hash function:

Definition 4.6 (TCR). Consider a family of hash functions $TCR_k : A_k \rightarrow B_k$. We define the TCR advantage of an attacker \mathcal{A} as follows:

$$Adv_{\mathcal{A}}^{TCR}(k) = \Pr \left[\begin{array}{c} x \neq y \wedge \\ TCR_k(x) = TCR_k(y) \end{array} : \begin{array}{c} x \xleftarrow{\$} A_k \\ y \xleftarrow{\$} \mathcal{A}(1^k, x) \end{array} \right].$$

A hash function TCR is target collision resistant if every PPT attacker \mathcal{A} has negligible advantage.

We will write TCR in place of TCR_k if the hash function family is implicit by context.

4.2 The Hybrid Cramer-Shoup Encryption Scheme

The hybrid Cramer-Shoup encryption scheme is defined using a group \mathbb{G} with generator g_1 of prime order q and a target collision-resistant hash function $TCR : \mathbb{G}^2 \rightarrow \mathbb{Z}_q$. The scheme was famously proven IND-CCA2 secure by Cramer and Shoup in 1998 [10]. We present the scheme as a KEM in Figure 21. The scheme has a symmetric key-space \mathbb{G} and so has to be used with a DEM with key-space \mathbb{G} . A DEM can with key space \mathcal{K} can be converted into a DEM with key space \mathbb{G} by using a smooth hash function⁷ $H : \mathbb{G} \rightarrow \mathcal{K}$.

<i>Gen</i> (1^k):	<i>Encap</i> (pk):	<i>Decap</i> (C):
$w \xleftarrow{\$} \mathbb{Z}_q; g_2 \leftarrow g_1^w$	$r \xleftarrow{\$} \mathbb{Z}_q$	Parse C as (c_1, c_2, π)
$x, y, z \xleftarrow{\$} \mathbb{Z}_q$	$c_1 \leftarrow g_1^r; c_2 \leftarrow g_2^r$	$t \leftarrow TCR(c_1, c_2)$
$e \leftarrow g_1^x; f \leftarrow g_1^y; h \leftarrow g_1^z$	$t \leftarrow TCR(c_1, c_2)$	If $c_2 = c_1^t$ and $\pi = c_1^{x+t+y}$
$pk \leftarrow (g_1, g_2, h, e, f)$	$\pi \leftarrow e^{rt} f^r$	Output c_1^w
$sk \leftarrow (w, x, y, z)$	$C \leftarrow (c_1, c_2, \pi)$	Else
Output (pk, sk)	$K \leftarrow h^r$	Output \perp
	Output (C, K)	

Fig. 21. The hybrid Cramer-Shoup encryption scheme (CS1b).

In accordance with the strategy in Section 3.4, we prove that this KEM is PA1+ and simulatable. Recall that, according to Remark 2.10, we are only required to show that the KEM is simulatable for attackers which obtain a single challenge ciphertext. In order to evaluate the simulatability of the Cramer-Shoup KEM, we need to introduce a new definition of security (which is very similar to the IND-R notion of security for symmetric encryption schemes)⁸. The IND-R security notion for a KEM is defined using the two games given in Figure 22.

⁷ A smooth hash function has the property that the distribution of $H(x)$, for $x \xleftarrow{\$} \mathbb{G}$, is computationally indistinguishable from the uniform distribution on \mathcal{K}

⁸ The original proof that the Cramer-Shoup encryption scheme is simulatable [12] has a (minor) flaw. The proof does not “reset” the decryption oracle to its correct operation at the end of proof. We avoid

$Encap'(pk, m):$ $C \xleftarrow{\$} \mathcal{C}$ $K \xleftarrow{\$} \mathcal{K}$ Output (C, K)	$EXPT_{\mathcal{A}}^{IND-R-1}(k):$ $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$ $\omega \xleftarrow{\$} \mathcal{A}_1^{Decap(sk, \cdot)}(1^k, pk)$ $(C^*, K^*) \xleftarrow{\$} Encap(pk)$ $b' \xleftarrow{\$} \mathcal{A}_2^{Decap(sk, \cdot)}(C^*, K^*, \omega)$ Return b'	$EXPT_{\mathcal{A}}^{IND-R-0}(k):$ $(pk, sk) \xleftarrow{\$} \mathcal{G}(1^k)$ $\omega \xleftarrow{\$} \mathcal{A}_1^{Decap(sk, \cdot)}(1^k, pk)$ $(C^*, K^*) \xleftarrow{\$} Encap'(pk)$ $b' \xleftarrow{\$} \mathcal{A}_2^{Decap(sk, \cdot)}(C^*, K^*, \omega)$ Return b'
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Fig. 22. The IND-R-CCA2 security notion for KEMs. In G_1 and G_2 the attacker \mathcal{A}_2 is forbidden from querying the decryption oracle on the ciphertext C^* .

Definition 4.7 (IND-R-CCA2). A KEM $\Pi = (Gen, Encap, Decap)$ is IND-R-CCA2 secure if for every PPT attacker $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ the advantage

$$Adv_{\mathcal{A}}^{IND-R}(k) = |\Pr[EXPT_{\mathcal{A}}^{IND-R-1}(k) = 1] - \Pr[EXPT_{\mathcal{A}}^{IND-R-0}(k) = 1]|$$

is negligible.

Cramer and Shoup (essentially) proved the following theorem [10]:

Theorem 4.8 (Cramer-Shoup). Suppose the DDH problem is hard in the group \mathbb{G} and the hash function TCR is target collision resistant. Then the Cramer-Shoup KEM is IND-R-CCA2 secure

There is a clear relationship between IND-R-CCA2 security and simulatability. A simple hybrid argument can show that the following:

Theorem 4.9. If a KEM Π is IND-R-CCA2 secure and both \mathcal{C} and \mathcal{K} are computationally/statistically simulatable (as sets), then Π is computationally/statistically simulatable (as a KEM).

Thus, we can easily conclude:

Corollary 4.10. If \mathbb{G} is computationally/statistically simulatable on which the DDH problem is hard, and TCR is target collision resistant, then the Cramer-Shoup KEM is computationally/statistically simulatable.

Proof Since \mathbb{G} is computationally/statistically simulatable, we have that both $\mathcal{C} = \mathbb{G}^3$ and $\mathcal{K} = \mathbb{G}$ are computationally/statistically simulatable (by Lemma 2.4). Hence, by a combination of Theorem 4.8 and Theorem 4.9, we have that the Cramer-Shoup KEM is computationally/statistically simulatable. \square

4.3 The Cramer-Shoup KEM is PA1+ Plaintext Aware

The proof that the Cramer-Shoup KEM is simulatable is heavily based on the Cramer and Shoup proof that it is IND-CCA2 secure. We introduce new techniques to prove that the Cramer-Shoup KEM is PA1+.⁹

the potential for this error by noting that almost all the required techniques needed to prove that the Cramer-Shoup KEM is simulatable are present in the original proof that the scheme is IND-CCA2 secure [10]. Hence, in this section we will merely point out the required extra steps.

⁹ The original proof of this result claimed that it was sufficient for the group \mathbb{G} to be computationally simulatable. This is incorrect: we require that the group \mathbb{G} is statistically simulatable. This error is relatively minor in practice as all the common groups which are used for cryptography are statistically simulatable (see Section 4.1). The original proof also failed to deal effectively with the differences between the DHK extractor \mathcal{B}^* and the PA1+ plaintext extractor \mathcal{A}^* in the way they deal with the random input to their respective underlying algorithms \mathcal{B} and \mathcal{A} : the DHK extractor \mathcal{B}^* requires all random bits of \mathcal{B} to be known in advance, while a PA1+ extractor \mathcal{A}^* allows for the possibility that fresh random bits will be generated by \mathcal{A} after \mathcal{A}^* 's execution.

Theorem 4.11. *Suppose the group \mathbb{G} is a statistically simulatable group on which the DHK assumption holds. Then the Cramer-Shoup KEM is PA1+ plaintext aware.*

Proof Let (Gf, Gf^{-1}, ℓ) be a simulator for the group \mathbb{G} . Let \mathcal{A} be an arbitrary PPT PA1+ ciphertext creator. We define a DHK attacker \mathcal{B} in Figure 23. Since the DHK assumption holds on \mathbb{G} , there exists a DHK extractor \mathcal{B}^* for \mathcal{B} , and we use this to define a plaintext extractor \mathcal{A}^* in Figure 23. One of the key points about the plaintext extractor \mathcal{A}^* is that for every ciphertext C submitted to the decryption oracle, the plaintext extractor runs the DHK extractor \mathcal{B}^* from scratch over all of the ciphertexts submitted to the decryption oracle up to this point. This is to cope with fact that the plaintext extractor \mathcal{A}^* is forced to guess the future randomness oracle queries RLIST' ; if the DHK extractor \mathcal{B}^* is run as a stateful algorithm in the obvious way (with one execution of \mathcal{B}^* for each execution of \mathcal{A}^*) then we cannot argue that \mathcal{B}^* 's responses are correct since the list of random coins RLIST' generated in the previous execution is unlikely to be correct. Thus, the proof of this theorem revolves centrally around an interesting argument that \mathcal{B}^* 's responses are identical when it is repeatedly run on the same ciphertext in different executions of the plaintext extractor \mathcal{A}^* (and therefore when it potentially has different internal states).

$\mathcal{B}^{\mathcal{O}}(1^k, g, a):$ $g_1 \leftarrow g; g_2 \leftarrow a$ $r_e, r_f, r_h \xleftarrow{\$} \{0, 1\}^\ell$ $e \leftarrow Gf(r_e); f \leftarrow Gf(r_f); h \leftarrow Gf(r_h)$ $pk \leftarrow (g_1, g_2, e, f, h)$ $R[\mathcal{A}] \xleftarrow{\$} \{0, 1\}^{t(k)}$ $x \leftarrow \mathcal{A}^{\mathcal{O}_D, \mathcal{O}_R}(1^k, pk)$ If \mathcal{A} queries \mathcal{O}_R $\rho \xleftarrow{\$} \{0, 1\}$ Return ρ If \mathcal{A} queries $\mathcal{O}_D(C)$ Parse C as (c_1, c_2, π) Query $r \leftarrow \mathcal{O}(c_1, c_2)$ $t \leftarrow \text{TCR}(c_1, c_2)$ If $c_1 = g_1^r$ and $c_2 = g_2^r$ and $\pi = e^{rt} f^r$ Return h^r Else Return \perp Output 0	$\mathcal{A}^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST}):$ On first invocation $\text{DLIST} \xleftarrow{\$} \varepsilon$ $n_D \leftarrow 0$ Else $n_D \leftarrow n_D + 1$ $r_e \leftarrow Gf^{-1}(e); r_f \leftarrow Gf^{-1}(f); r_h \leftarrow Gf^{-1}(h)$ $\text{RLIST}' = \{0, 1\}^{q_R - \text{RLIST} }$ $R[\mathcal{B}] \leftarrow r_e \ r_f \ r_h \ R[\mathcal{A}] \ \text{RLIST} \ \text{RLIST}'$ Reset \mathcal{B}^* (i.e. delete its internal state) Append C to DLIST For $j = 0, 1, \dots, n_D$ $C_j \leftarrow \text{DLIST}[j]$ Parse C_j as (c_1, c_2, π) $r_j \leftarrow \mathcal{B}^*(g_1, g_2, c_1, c_2, R[\mathcal{B}])$ $t \leftarrow \text{TCR}(c_1, c_2)$ If $c_1 = g_1^r$ and $c_2 = g_2^r$ and $\pi = e^{rt} f^r$ $m_j \leftarrow h^r$ Else $m_j \leftarrow \perp$ Output m_{n_D}
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Fig. 23. The DHK algorithm \mathcal{B} and the plaintext extractor \mathcal{A}^* . Since \mathcal{A} runs in strict polynomial-time attacker, there exists a polynomial bound $t(k)$ on the running time of \mathcal{A} , and so we may assume that $R[\mathcal{A}]$ is $t(k)$ -bits long. Recall that \mathcal{A}^* and \mathcal{B}^* run as stateful algorithms. DLIST contains a list of ciphertexts which have been submitted to the decryption oracle. The counter n_D gives the index for the current ciphertext C in DLIST .

Let D be an arbitrary polynomial-time distinguisher algorithm and W_i be the event that $D(x)$ outputs 1 in game G_i .

Game G_1 : G_1 is the $\text{EXPT}_{\mathcal{A}, \mathcal{A}^*, D}^{\text{FAKE-PA1+}}(k)$ game.

Game G_2 : Note that G_1 does not make use of the private key. Hence, the group elements e, f, h are uniformly distributed in \mathbb{G} . In G_2 these elements are generated by choosing $\tilde{r}_e, \tilde{r}_f, \tilde{r}_h \xleftarrow{\$} \{0, 1\}^\ell$ and setting $e \leftarrow Gf(\tilde{r}_e)$, $f \leftarrow Gf(\tilde{r}_f)$, and $h \leftarrow Gf(\tilde{r}_h)$. This is a distributional game hopping step (see Section 2.1). Let U be the uniform distribution on \mathbb{G} and Y be the distribution of $Gf(\tilde{r})$ for $\tilde{r} \xleftarrow{\$} \{0, 1\}^\ell$. So $|\Pr[W_2] - \Pr[W_1]| \leq \Delta[U, Y]$ which is negligible by the assumption that \mathbb{G} is statistically simulatable.

Game G_3 : Note that in G_2 the plaintext extractor \mathcal{A}^* re-computes the values r_e, r_f, r_h as $Gf^{-1}(Gf(\tilde{r}_e)), Gf^{-1}(Gf(\tilde{r}_f)), Gf^{-1}(Gf(\tilde{r}_h))$. G_3 alters \mathcal{A}^* to use the original values $\tilde{r}_e, \tilde{r}_f, \tilde{r}_h$. This is a distributional game hopping step. Let U be the uniform distribution on $\{0, 1\}^\ell$ and let Y be the distribution of $Gf^{-1}(Gf(r))$ for $r \xleftarrow{\$} \{0, 1\}^\ell$. So $|\Pr[W_3] - \Pr[W_2]| \leq \Delta[U, Y]$ which is negligible by the assumption that \mathbb{G} is statistically simulatable.

Game G_4 : G_4 responds to decryption oracle using the plaintext extractor \mathcal{A}_4^* rather than \mathcal{A}^* . \mathcal{A}_4^* is almost identical to \mathcal{A}^* except that it no longer uses \mathcal{B}^* but instead uses computes the Diffie-Hellman value r exactly (even though this makes a \mathcal{A}_4^* a non-polynomial-time algorithm). The complete description is given in Figure 24.

$\mathcal{A}_4^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$: Parse C as (c_1, c_2, π) If there exists no r such that $c_1 = g_1^r$ and $c_2 = g_2^r$ Return \perp Find r such that $c_1 = g_1^r$ and $c_2 = g_2^r$ $t \leftarrow \text{TCR}(c_1, c_2)$ If $\pi = e^{rt} f^r$ Return h^r Else Return \perp	$\mathcal{A}_7^*(1^k, pk, C, R[\mathcal{A}], \text{RLIST})$: Parse C as (c_1, c_2, π) If $c_2 \neq c_1^w$ Return \perp $t \leftarrow \text{TCR}(c_1, c_2)$ If $\pi = c_1^{xt+y}$ Return c_1^z Else Return \perp
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Fig. 24. The plaintext extractors \mathcal{A}_4^* and \mathcal{A}_7^* used in G_4 and G_7 .

Let E be the event that the decryption oracle returns a different response in G_3 and G_4 and let E_i be the event that this *first* occurs on the i -th decryption oracle query. Suppose that the i -th decryption oracle query occurs on a ciphertext (c_1, c_2, π) . If (g_1, g_2, c_1, c_2) is not a Diffie-Hellman tuple, then the decryption oracle will return \perp in G_3 and G_4 . Hence, E_i only occurs if \mathcal{A} submits a ciphertext to the decryption oracle for which (g_1, g_2, c_1, c_2) is a valid Diffie-Hellman tuple but for which \mathcal{A}^* fails to compute a value r_i such that $c_1 = g_1^{r_i}$ and $c_2 = g_2^{r_i}$.

We will bound $\Pr[E_i]$. Consider the i -th decryption oracle query. \mathcal{A}^* computes r_i by simulating an execution of the DHK game using \mathcal{B} and \mathcal{B}^* for i DHK oracle queries. This simulation doesn't actually run \mathcal{B} but supplies \mathcal{B}^* with the ciphertexts that \mathcal{B} would have computed using DLIST. Within this execution, let E'_j be the event that \mathcal{B}^* fails to determine the correct value \tilde{r}_j when queried with the j -th ciphertext (i.e. the j -th DHK oracle query is a pair (c_1, c_2) such that $c_1 = g_1^r$ and $c_2 = g_2^r$ but \mathcal{B}^* fails to return the correct value of r).

- If any E'_j occurs, with $1 \leq j < i$, then we cannot determine whether \mathcal{B}^* will correctly determine the value r_i that \mathcal{A}^* requires. This is because the internal state of \mathcal{B}^* would be consistent with an execution of \mathcal{B} which computed an incorrect decryption of the j -th ciphertext. Hence, the list of ciphertexts DLIST may be inconsistent with the ciphertexts output by the algorithm \mathcal{B} that \mathcal{A}^* is trying to simulate. (Recall that E_i represents the first time the decryption oracle incorrectly decrypts a ciphertext; hence, all previous ciphertexts must have been decrypted correctly.)
- If E'_i occurs, then the value r_i computed by \mathcal{A}^* is incorrect by the definition of E'_i .

Thus, we can conclude

$$\Pr[E_i] \leq \Pr[E'_1 \vee E'_2 \vee \dots \vee E'_{i-1} \vee E'_i].$$

However, by the definition of the DHK game, this value is bounded by $\text{Adv}_{\mathcal{B}, \mathcal{B}^*}^{\text{DHK}}(k)$. Thus, $\Pr[E_i] \leq \text{Adv}_{\mathcal{A}, \mathcal{A}^*}^{\text{DHK}}(k)$ and so $|\Pr[W_4] - \Pr[W_3]| \leq \Pr[E] \leq q_D \text{Adv}_{\mathcal{A}, \mathcal{A}^*}^{\text{DHK}}(k)$ which is negligible.

Game G_5 : G_5 changes \mathcal{A}_4^* so that it computes the values r_e, r_f, r_h as $Gf^{-1}(e), Gf^{-1}(f), Gf^{-1}(h)$ respectively. This essentially “undoes” the change made in G_3 and by the same argument we have that $|\Pr[W_5] - \Pr[W_4]|$ is negligible since \mathbb{G} is statistically random-set simulatable. (Note

that we require *statistical* random-set simulatability here as the extractor is not polynomial-time and so there is no guarantee that computational random-set simulatability would suffice.)

Game G_6 : G_6 changes the distribution of e, f, h so that they are generated as $e, f, h \stackrel{\$}{\leftarrow} \mathbb{G}$ rather than by using Gf . This “undoes” the change made in G_2 and by the same argument we have that $|\Pr[W_6] - \Pr[W_5]|$ is negligible since \mathbb{G} is statistically image-set simulatable.

Game G_7 : G_7 changes the distribution of g_2, e, f, h and uses a plaintext extractor \mathcal{A}_7^* given in Figure 24. The elements g_2, e, f, h are computed by selecting $w, x, y, z \stackrel{\$}{\leftarrow} \mathbb{Z}_q$ and setting $g_2 \leftarrow g_1^w$, $e \stackrel{\$}{\leftarrow} g_1^x$, $f \stackrel{\$}{\leftarrow} g_1^y$, $h \stackrel{\$}{\leftarrow} g_1^z$. Obviously, this doesn’t change their distribution, so we concentrate on showing that the action of \mathcal{A}_7^* is identical to the version of \mathcal{A}_4^* used in G_6 . However, this can trivially be seen to be true by noting that $c_2 = c_1^w$ if and only if there exist an r such that $c_1 = g_1^r$ and $c_2 = g_2^r$, and that if $c_1 = g_1^r$ then $e^{rt} f^r = c_1^{rt} c_1^r$ and $h^r = c_1^z$. Thus, $\Pr[W_7] = \Pr[W_6]$.

But an examination of \mathcal{A}_7^* shows that it is functionally identical to the real decryption algorithm $\mathcal{D}(sk, \cdot)$. Thus, the event W_7 is equivalent to the event $\text{EXPT}_{\mathcal{A}, \mathcal{D}}^{\text{REAL-PA1}^+}(k) = 1$. And so, $\text{Adv}_{\mathcal{A}, \mathcal{D}}^{\text{PA1}^+}(k) \leq |\Pr[W_1] - \Pr[W_7]|$ is negligible and so the Cramer-Shoup KEM is PA1+. \square

Corollary 4.12. *Suppose \mathbb{G} is a statistically simulatable group on which both the DDH problem is hard and the DHK assumption holds and that TCR is a target collision resistant hash function. Then the Cramer-Shoup KEM is PA2 plaintext aware.*

Therefore, we have that the hybrid Cramer-Shoup encryption scheme (with $\mathcal{M} = \{0, 1\}^n$) is PA2 plaintext aware. (Recall that the condition that $\mathcal{M} = \{0, 1\}^n$ is required in order to prove that the DEM is simulatable. It is also a necessary condition for an encryption scheme to be PA2 plaintext aware.)

5 Conclusions and Open Problems

This paper gave the first proof strategy for proving PA2 plaintext awareness and used this strategy to prove that the hybrid Cramer-Shoup encryption scheme is PA2 plaintext aware. At the heart of this strategy are the notions of simulatability and PA1+ plaintext awareness. Both are strong requirements: a scheme which is simulatable is necessarily IND-CCA2 secure and a proof of PA1+ plaintext awareness seems to require strong extractor assumptions like the Diffie-Hellman Knowledge (DHK) assumption. Nevertheless, the fact that PA2 plaintext aware encryption schemes can be proven to exist at all is theoretically interesting and the fact that the Cramer-Shoup encryption scheme is PA2 plaintext aware gives some explanation as why the scheme achieves IND-CCA2 security. The approach demonstrated here has been used to show that all Cramer-Shoup hash-proof encryption schemes [7] and Kurosawa-Desmedt hash-proof encryption schemes [7, 18] are also plaintext aware (under certain extractor-based assumptions).

There are many open problems in this area. The most important ones are:

- Is it possible to show any other schemes are plaintext aware? All the schemes that are currently known to be PA2 plaintext aware are of the “hash-proof” variety. It would be interesting to see if any other schemes could achieve this level of security.
- Is it possible to use plaintext-aware encryption to reason about protocols and attack strategies? This approach was taken in the work of Di Raimondo, Gennaro and Krawczyk [22] and Ventre and Visconti [29], but it may be useful in other scenarios, e.g. in the analysis of protocols using formal methods.
- Is it possible to develop a strategy for proving PA2 plaintext awareness that does not rely on simulatable encryption schemes? Since one of the major motivations for the study of plaintext-aware encryption is its use in proving IND-CCA2 security, the reliance of the strategy on simulatable encryption schemes, which are already necessarily IND-CCA2 secure, is a disadvantage.
- Is it possible to prove that a scheme is PA1/PA2 plaintext aware without the use of an extractor-based assumption like the DHK assumption? Alternatively, can it be shown that plaintext awareness cannot be proven under any non-interactive assumption?

It is also interesting to examine the common strategies for proving IND-CCA2 security which do not explicitly rely on plaintext awareness. After all, these strategies have to respond to “trivial” decryption oracle queries (where the adversary submits a ciphertext to the decryption oracle for which it already knows the underlying message) too. The typical strategy seems to be a “common random string” version of plaintext awareness in which a common random string is published as part of the system parameters. This common random string is used in a manner similar to a public key to control part of the encryption process. The security proof demonstrates that:

1. The common random string can be generated with a “trapdoor” (and no polynomial-time adversary will notice the difference between a common random string and a common random string generated with a trapdoor).
2. The encryption algorithm can be altered so that the elements of the ciphertext which depend on the common random string can be computed from public information, such as the other parts of the ciphertext (and, perhaps, the trapdoor). Most notably, the elements of the ciphertext which depend on the common random string do not (directly) depend on the value of the message being encrypted.
3. The trapdoor allows a simulator to decrypt ciphertexts in a space $\mathcal{C}' \subseteq \mathcal{C}$.
4. It is computationally infeasible for a polynomial-time adversary to submit a ciphertext $C \in \mathcal{C} \setminus \mathcal{C}'$ to the decryption oracle. The trapdoor then provides a version of plaintext awareness: the trapdoor can be used to produce a “plaintext extractor” which can answer the adversary’s encryption oracle queries without using the private key of the recipient.
5. The encryption algorithm is IND-CPA secure against adversaries (who are given the trapdoor).

This “common random string” notion of plaintext awareness relates to “full” plaintext awareness in a manner similar to the way in which non-interactive zero-knowledge (NIZK) proofs relate to zero-knowledge protocols. The fact that this strategy allows proofs of IND-CCA2 security without requiring extractor assumptions is conceptually analogous to the fact that NIZK proof allow one party to prove a theorem to another party using a single message whereas full zero-knowledge protocols require at least two rounds of interaction between parties.

It is also interesting to examine other ways in which the extraction methodology can be applied. One interesting extension is the concept of secret-key awareness, which moves the knowledge assumption from the ciphertext to the public key, i.e. a scheme is secret-key aware if it is infeasible for any PPT algorithm to create a public key without “knowing” the underlying private key [1]. This has applications to proving complete non-malleability [13] and may have applications towards protocol analysis.

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