Tracing Monoidal Man in the Middle (with hindsight)

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Outline

Category of protocols

Trace monad

Future traces

Outline

Category of protocols

- Arities
- Clones
- Programs
- Protocols
- Protocol analysis

Trace monad

Future traces

Category of arities

$$
\begin{aligned}
|\mathbb{A}| &= \left\{n = \{0, 1, \ldots, n-1\} \mid n \in \mathbb{N}\right\} \\
\mathbb{A}(m, n) &= \left\{ (x_0, \ldots, x_{m-1})\langle x_{f(0)}, \ldots, x_{f(n-1)}\rangle \mid f : n \to m\right\} \big/ \alpha
\end{aligned}
$$

Category of arities

Remark

$\mathbb{A}$ is the free strict cartesian category generated by 1.

Algebraic theories

- An algebraic theory is a pair $\mathcal{T} = \langle \Sigma_{\mathcal{T}}, E_{\mathcal{T}} \rangle$ where
  - $\Sigma = \Sigma_{\mathcal{T}}$ is a signature, and
  - $E = E_{\mathcal{T}}$ is a set of equations

Algebraic theories

- An algebraic theory is a pair $\mathcal{T} = \langle \Sigma_{\mathcal{T}}, E_{\mathcal{T}} \rangle$ where
  - $\Sigma = \Sigma_{\mathcal{T}}$ is a signature, and
  - $E = E_{\mathcal{T}}$ is a set of equations
- Algebraic operations $\varphi(x_1, \ldots, x_n)$ are generated from Σ modulo E.
- A $\mathcal{T}$-algebra assigns a map $\varphi \; : \; X^n \to X$ for each n-ary algebraic operation φ.

Lawvere clones (à la Milner)

$$\begin{aligned} |\mathbb{C}_{\mathcal{T}}| &= \mathbb{N} \\ \mathbb{C}_{\mathcal{T}}(m,n) &= \Big\{ (x_0, \ldots, x_{m-1}) \langle \varphi_0, \ldots, \varphi_{n-1} \rangle \Big\} \Big/ \alpha \end{aligned}$$

Programming languges

- A programming language $\mathcal{L}$ generates programs P with inputs $(x_0, \ldots, x_{m-1})$ and outputs $\langle s_0, \ldots, s_{n-1} \rangle$.
- Semantics of $\mathcal{L}$ assigns a map $P : \prod_{i \in m} X_i \to \prod_{j \in n} S_j$ for each program P with m inputs and n outputs.

Category of programs (untyped)

$$\begin{aligned} |\mathbb{P}_{\mathcal{L}}| &= \mathbb{N} \\ \mathbb{P}_{\mathcal{L}}(m,n) &= \Big\{ (x_0, \ldots, x_{m-1}) [P] \langle s_0, \ldots, s_{n-1} \rangle \Big\} \Big/ \alpha \end{aligned}$$

(Functorial semantics)

$$\frac{\text{equational theories}}{\text{algebras}} = \frac{\text{programming languages}}{\text{algebras} \subsetneq \text{coalgebras}}$$

Category of programs

Remark

- typing $\rightsquigarrow$ extended arities

Category of programs

Remark

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- sequential composition $\leftrightsquigarrow$ arrow composition

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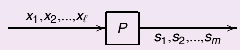
Category of programs

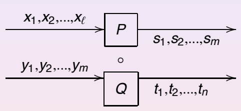
Remark

- typing $\leftrightsquigarrow$ extended arities
- sequential composition $\leftrightsquigarrow$ arrow composition
- parallel composition $\leftrightsquigarrow$ monoidal structure
- program loops $\leftrightsquigarrow$ trace structure

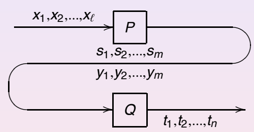
Category of programs: String diagrams



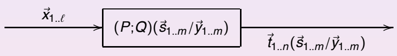
Category of programs: Composition



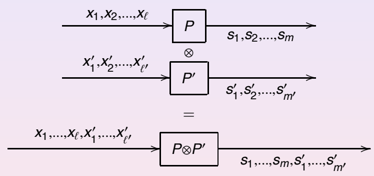
Category of programs: Composition



Category of programs: Composition

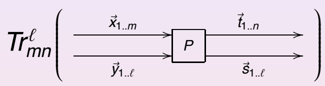


Category of programs: Tensor

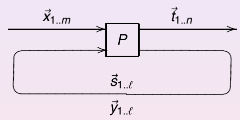


where $\vec{x}_{1..\ell} \cap \vec{x'}_{1..\ell'} = \emptyset$

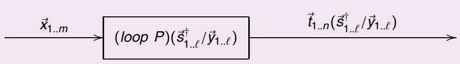
Category of programs: Trace



Category of programs: Trace



Category of programs: Trace



Category of programs: Trace

But verifying the trace structure is a lot of work!

Category of programs: Trace

But verifying the trace structure is a lot of work!

If the traces are OK, we model protocols as follows:

Category of protocols

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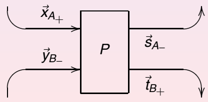
$$
\begin{aligned}
|\mathbb{J}| &= \left\{A = \langle A_+, A_-\rangle \in |\mathbb{P}|^2\right\}\\
\mathbb{J}(A,B) &= \{A_+ \otimes B_- \to A_- \otimes B_+ \text{ in } \mathbb{P}\}
\end{aligned}
$$

$$
\begin{pmatrix} \vec{x}_{A_+} \\ \vec{y}_{B_-} \end{pmatrix} \left[\, P \,\right] \left\langle \begin{matrix} \vec{s}_{A_+} \\ \vec{t}_{B_-} \end{matrix} \right\rangle = \begin{pmatrix} x_1 & x_2 & \dots & x_{A_+} \\ y_1 & y_2 & \dots & y_{B_-} \end{pmatrix} \left[\, P \,\right] \left\langle \begin{matrix} s_1 & s_2 & \dots & s_{A_-} \\ t_1 & t_2 & \dots & t_{B_+} \end{matrix} \right\rangle
$$



Composition of protocols

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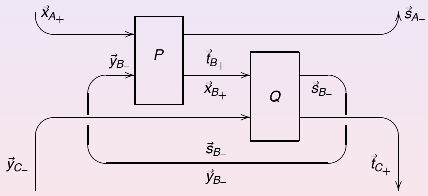

Categorical protocol analysis

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A protocol is

- specified by giving
  1. local observations at the interfaces
  2. a (security) requirement on the interactions
- represented as a morphism $A \xrightarrow{p} B$ in $\mathbb{J}$
- analyzed by enumerating

  Man-in-the-Middle: nontrivial decompositions
  $A \to X \to B$

  Chosen Protocol Attack: leaking compositions
  $A \to B \to X$

Outline

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Category of protocols

Trace monad

Local monoidal categories

Normal traces

Loop categories

Free normal traces

Future traces

Local monoids

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Definition

Let $\mathbb{M} \times \mathbb{M} \xrightarrow{\cdot} \mathbb{M} \xleftarrow{\iota} 1$ be a commutative monoid.

- $o \in \mathbb{M}$ is *zero* if $ou = o$ for all $u \in \mathbb{M}$
- $s \in \mathbb{M}$ is a *nilpotent* if $s^n = o$ for some $n \in \mathbb{N}$
- $s \in \mathbb{M}$ is *regular* if it is not nilpotent.
- $\mathbb{M}$ is *local* if all regular elements are invertible

Local monoidal categories

Definition

Let $\mathbb{C}\times\mathbb{C}\xrightarrow{\otimes}\mathbb{C}\xleftarrow{I}1$ be a small symmetric monoidal category. Then

- $\mathbb{I}=\mathbb{C}(I,I)$ is a commutative monoid
- $\mathbb{I}\times\mathbb{C}(A,B)\overset{\cdot}{\to}\mathbb{C}(A,B)$ is $\mathbb{I}$-action
- $A\xrightarrow{o_{AB}}B$ is *zero* if $o\cdot o_{AB}=o_{AB}$
- $\mathbb{C}$ is a *local* monoidal category if $\mathbb{I}$ is a local monoid.

Graded monoidal categories

Definition

A small strict symmetric monoidal category

$$\mathbb{C}\times\mathbb{C}\xrightarrow{\otimes}\mathbb{C}\xleftarrow{I}1$$

is *graded* by a monoid homomorphism

$$(\mathbb{C},\otimes,I)\xrightarrow{|-|}(\mathbb{I},\circ,\mathrm{id})$$

where $\mathbb{I}=\mathbb{C}(I,I)$.

Graded monoidal categories

Remark

Every traced monoidal category is graded by

$$|U| \quad = \quad Tr^U(\mathrm{id}_U)$$

Normal traces

Definition

A *normal* trace structure over a monoidal category $\mathbb{C}$ is a family of operators

$$\frac{f\ :\ A\otimes U\to B\otimes U}{Tn^U_{AB}(f)\ :\ A\to B}$$

satisfying the Joyal-Street-Verity axioms and also:

$$Tn^U_{AB}\left(A\otimes U\xrightarrow{g\otimes U}B\otimes U\right) \quad = \quad \begin{cases} A\xrightarrow{g}B & \text{if } U \text{ is regular}\\ A\xrightarrow{o}B & \text{otherwise}\end{cases}$$

Trace decomposition

Proposition 1

Any trace structure over a local monoidal category $\mathbb{C}$ decomposes:

$$Tr^U(f) \quad = \quad |U|\cdot Tn^U(f)$$

Free normal traces

Proposition 2

Normal traces are monadic over local monoidal categories.

Loop categories

Definition

The loop category over a small local monoidal category $\mathbb{C}$ is

$$\begin{aligned} |\mathbb{C}^{\circlearrowleft}| &= |\mathbb{C}| \\ \mathbb{C}^{\circlearrowleft}(A,B) &= \oint_{U\in|\mathbb{C}|} \mathbb{C}(A\otimes U, B\otimes U) \end{aligned}$$

Loop categories

Definition

The loop category over a small local monoidal category $\mathbb{C}$ is

$$\begin{aligned} |\mathbb{C}^{\circlearrowleft}| &= |\mathbb{C}| \\ \mathbb{C}^{\circlearrowleft}(A,B) &= \left(\sum_{U\in|\mathbb{C}|} \mathbb{C}(A\otimes U, B\otimes U) \times \mathbb{I}^*\right)\Big/\sim \end{aligned}$$

Loop categories

...where $\sim$ extends the coend equivalence...

$$\begin{tikzcd} & \mathbb{C}(A\otimes U, B\otimes U) \arrow[dr, dashed] & \\ \mathbb{C}(A\otimes V, B\otimes U) \arrow[ur, "(-)\circ(A\otimes u)"] \arrow[rr, dashed] \arrow[dr, "(B\otimes u)\circ(-)"'] & & \mathbb{C}^{\circlearrowleft}(A,B) \\ & \mathbb{C}(A\otimes V, B\otimes V) \arrow[ur, dashed] & \end{tikzcd}$$

Loop categories

...where $\sim$ extends the coend equivalence...

$$\frac{(B\otimes u)\circ f}{s} \sim \frac{f\circ(A\otimes u)}{s}$$

$$\begin{tikzcd} & A\otimes U \arrow[dl, "f"'] \\ B\otimes V \arrow[dr, "B\otimes u"'] & \\ & B\otimes U \end{tikzcd} \qquad \begin{tikzcd} A\otimes V \arrow[dr, "A\otimes u"] & \\ & A\otimes U \arrow[dl, "f"] \\ B\otimes V & \end{tikzcd}$$

Loop categories

...by

$$\frac{f\otimes c_U}{s} \sim \frac{f\otimes U}{s}$$

$$\begin{tikzcd} A\otimes U\otimes U \arrow[d, "f\otimes c_U"'] \\ B\otimes U\otimes U \end{tikzcd} \qquad \begin{tikzcd} A\otimes U \arrow[d, "f\otimes U"] \\ B\otimes U \end{tikzcd}$$

Loop categories

...and

$$\frac{f}{s} \sim \frac{g}{t}$$

$$\Updownarrow$$

$$\exists uv\in\mathbb{I}^*.\ u\circ f = v\circ g \ \wedge\ u\circ s = v\circ t$$

Composition

Given

- $f \in \mathbb{C}^{\circlearrowleft}(A, B)$ as $A \otimes U \xrightarrow{f_0/f_1} B \otimes U$, and
- $g \in \mathbb{C}^{\circlearrowleft}(B, C)$ as $B \otimes V \xrightarrow{g_0/g_1} C \otimes V$,

the composite

- $f \circ g \in \mathbb{C}^{\circlearrowleft}(A, C)$ can be viewed as

$$\begin{tikzcd}
A \otimes U \otimes V \arrow[r, "\frac{f_0 \otimes V}{f_1 \circ |V|}"'] & B \otimes U \otimes V \arrow[d, "B \otimes c"] & C \otimes U \otimes V \\
& B \otimes V \otimes U \arrow[r, "\frac{g_0 \otimes U}{g_1 \circ |U|}"'] & C \otimes V \otimes U \arrow[u, "C \otimes c"']
\end{tikzcd}$$

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Given

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A \otimes V \otimes U \arrow[u, "A \otimes c"'] & B \otimes V \otimes U \arrow[r, "\frac{g_0 \otimes U}{g_1 \circ |U|}"'] & C \otimes V \otimes U
\end{tikzcd}$$

Tensor

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Given

- $f \in \mathbb{C}^{\circlearrowleft}(A, B)$ as $A \otimes U \xrightarrow{f_0/f_1} B \otimes U$, and
- $h \in \mathbb{C}^{\circlearrowleft}(C, D)$ as $C \otimes V \xrightarrow{h_0/h_1} D \otimes V$,

the tensor product

- $f \otimes h \in \mathbb{C}^{\circlearrowleft}(A \otimes C, B \otimes D)$ can be viewed as

$$\begin{tikzcd}
A \otimes C \otimes U \otimes V \arrow[d, "A \otimes c \otimes V"] & B \otimes D \otimes U \otimes V \\
A \otimes U \otimes C \otimes V \arrow[r, "\frac{f_0 \otimes h_0}{f_1 \circ h_1}"'] & B \otimes U \otimes D \otimes V \arrow[u, "B \otimes c \otimes V"']
\end{tikzcd}$$

Normal trace

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$$\frac{f \;=\; \dfrac{(A \otimes U) \otimes V \xrightarrow{f_0} (B \otimes U) \otimes V}{f_1} \;\in\; \mathbb{C}^{\circlearrowleft}(A \otimes U, B \otimes U)}{Tn^U_{AB} f \;=\; \dfrac{A \otimes (U \otimes V) \xrightarrow{f_0} B \otimes (U \otimes V)}{f_1} \;\in\; \mathbb{C}^{\circlearrowleft}(A, B)}$$

Free normal traces

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Proposition 2 (continuation)

The free normal traces are given by the loop monad:

- 2-functor $\circlearrowleft : \mathcal{LM} \to \mathcal{LM}$
- unit functors

$$\begin{aligned} \eta_{\mathbb{C}} \;&:\; \mathbb{C} \to \mathbb{C}^{\circlearrowleft} \\ (A \xrightarrow{f} B) &\longmapsto \left[A \otimes I \xrightarrow{f \otimes I} B \otimes I\right]_{\sim} \end{aligned}$$

Free normal traces

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- evaluation functors

$$\mu_{\mathbb{C}} \;:\; \mathbb{C}^{\circlearrowleft\circlearrowleft} \to \mathbb{C}^{\circlearrowleft}$$

$$\left[\frac{\left[\dfrac{(A \otimes U) \otimes V \xrightarrow{f_0} (B \otimes U) \otimes V}{s}\right]_{\sim}}{t}\right]_{\sim} \longmapsto \left[\frac{A \otimes (U \otimes V) \xrightarrow{f_0} B \otimes (U \otimes V)}{s \circ t}\right]_{\sim}$$

Free normal traces

Corollary

$$\begin{aligned} |\mathbb{P}_{\mathcal{L}}| &= \mathbb{N} \\ \mathbb{P}_{\mathcal{L}}(m,n) &= \left\{ (x_0,\ldots,x_{m-1})[P]\langle s_0,\ldots,s_{n-1}\rangle \right\} \big/ \alpha \end{aligned}$$

is a normal traced monoidal category.

Outline

Category of protocols

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Future traces

- Weak distributivity
- Weak traces
- Weak interactions

Weakly distributive categories

Definition

A weakly distributive category $\mathbb{C}$ carries two (symmetric!) monoidal structures

$$\mathbb{C}\times\mathbb{C} \xrightarrow{\otimes} \mathbb{C} \xleftarrow{\top} 1$$

$$\mathbb{C}\times\mathbb{C} \xrightarrow{\oplus} \mathbb{C} \xleftarrow{\perp} 1$$

together with

$$A\otimes(B\oplus C) \xrightarrow{d} (A\otimes B)\oplus C \qquad \perp \xrightarrow{m} \top$$

coherently. . .

Weak traces

Definition

The normal trace operator over a weakly distributive category $\mathbb{C}$ is a family of operators

$$\frac{f \;:\; A\otimes U \to B\oplus U}{Tn^U_{AB} f \;:\; A \to B}$$

satisfying the well-typed versions of the familiar laws.[1]

[1] Vanishing can be weakened!

Weak interactions

Definition

Given a weakly distributive traced category $\mathbb{C}$, the induced interaction category $\mathcal{W}\mathbb{J} = \mathrm{Int}(\mathbb{C})$ consists of

$$\begin{aligned} |\mathcal{W}\mathbb{J}| &= \left\{ A = \langle A_+, A_-\rangle \in |\mathcal{C}|^2 \right\} \\ \mathbb{J}(A,B) &= \{ A_+\otimes B_- \to A_-\oplus B_+ \text{ in } \mathcal{C} \} \end{aligned}$$

where

$$\frac{A_+\otimes B_- \xrightarrow{f} A_-\oplus B_+ \qquad B_+\otimes C_- \xrightarrow{g} B_-\oplus C_+}{g\circ f = Tn^{B_-}_{A_+C_-}\left(A_+\otimes B_-\otimes C_- \xrightarrow{f\otimes C_-} A_-\oplus B_+\otimes C_- \xrightarrow{A\oplus g} A_-\oplus B_-\oplus C_+\right)}$$

$$\mathrm{id}_A = \left(A_+\otimes A_- \cong A_+\oplus\perp\otimes A_- \to A_+\oplus\top\otimes A_- \cong A_+\oplus A_-\right)$$

Weak interactions

Conjecture

$\mathbb{C} \to \mathrm{Int}(\mathbb{C})$ is the free $*$-autonomous weakly traced category.

Weak interactions

Conjecture

$\mathbb{C} \to \mathrm{Int}(\mathbb{C})$ is the free $*$-autonomous weakly traced category.

Question

What does this mean logically? Geometrically?