Way below security

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April 2009
MFPS XXV, Oxford
Session honoring Mike Mislove

Outline

Outline

Security is complicated

Crypto system

Given the types

- $\mathcal{M}$ of *plaintexts*
- $\mathcal{C}$ of *cyphertexts*
- $\mathcal{K}$ of *keys*
- $\mathcal{R}$ of *random seeds*

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Given the types

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a *crypto-system* is a triple of algorithms:

- key generation $\langle k, \overline{k} \rangle : \mathcal{R} \longrightarrow \mathcal{K} \times \mathcal{K}$,
- encryption $\mathsf{E} : \mathcal{R} \times \mathcal{K} \times \mathcal{M} \longrightarrow \mathcal{C}$, and
- decryption $\mathsf{D} : \mathcal{K} \times \mathcal{C} \longrightarrow \mathcal{M}$,

Security is complicated

Crypto system

. . . that together provide

- unique decryption:

$$\mathsf{D}(\overline{k}, \mathsf{E}(k, m)) \quad = \quad m$$

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Crypto system

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- unique decryption:

$$\mathrm{D}(\overline{k}, \mathrm{E}(k, m)) \quad = \quad m$$

- secrecy (IND-CCA):

$$\mathrm{Prob}\left(\begin{array}{l} c_0 \in \mathbb{A}_0,\ m = \mathrm{D}(\overline{k}, c_0), \\ m_0, m_1 \in \mathbb{A}_1(c_0, m),\ c \in \mathrm{E}(k, m_b) \\ c_1 \in \mathbb{A}_2(c_0, m, m_0, m_1, c^{\neq}),\ \widetilde{m} = \mathrm{D}(\overline{k}, c_1) \end{array} \vdash \right.$$
$$\left. b \in \mathbb{A}_3(c_0, m, m_0, m_1, c, c_1, \widetilde{m}) \right) \quad \leq \quad \frac{1}{2}$$

for any probabilistic algorithm $\mathbb{A} = \langle \mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \rangle$

Idea

Symbolically, the secret is guarded by a key

$$\forall \Theta.\ \ \Theta,\ \mathrm{E}(k, m) \vdash m \quad \Longrightarrow \quad \Theta \vdash k$$
$$\Updownarrow$$
$$k \quad \text{guards} \quad m \quad \text{in} \quad \mathrm{E}(k, m)$$

Idea

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$$k \quad \text{guards} \quad m \quad \text{in} \quad \mathrm{E}(k, m)$$

Does that mean that the key is way-below the secret?

$$\forall J.\ \ \bigsqcup J \sqsupseteq m \quad \Longrightarrow \quad J \sqsupseteq k$$
$$\Updownarrow$$
$$k \quad \ll \quad m$$

Outline

Information system of a protocol

Algebraic model

- algebra $\mathbb{T}$
  - equational presentation (Σ, E), generators V
- principals $\mathcal{W}$
  - $A \in \mathcal{W}$ owns $x \in \mathrm{V}_A$, a store, nonce or key
- fixed protocol run $\mathcal{Q}$
  - $A \in \mathcal{W}$ may send $t \in \mathbb{T}$, or receive into $x \in \mathrm{V}_A$

Information system of a protocol

Derivability

$$\Gamma \vdash \Theta \quad \Longleftrightarrow \quad \forall t \in \Theta\ \ \exists \varphi \in \Sigma^*\ \ \exists \vec{g} \subseteq \Gamma.\ \ \varphi(\vec{g}) \stackrel{E}{=} t$$

Consistent sets

$$\begin{array}{rcl} \Gamma^{\mathcal{Q}}_A & = & A\text{'s environment in } \mathcal{Q} \\ \mathrm{Con}^{\mathcal{Q}}_A & = & \{\Theta \in \wp_{fin}\mathbb{T} \mid \Gamma^{\mathcal{Q}}_A \vdash \Theta\} \\ \mathrm{Con}^{\mathcal{Q}} & = & \bigcup_{A \in \mathcal{W}} \mathrm{Con}^{\mathcal{Q}}_A \end{array}$$

Example: Encryption protocol

- $\Sigma = \{\mathsf{E}, \mathsf{D} : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}\}$
- $E = \{\mathsf{D}(x, \mathsf{E}(x, y)) = y\}$
- $\mathcal{Q} = \{A \xrightarrow{\mathsf{E}(k,m)} B\}$
  - $k \in \Gamma_X \iff X \in \{A, B\}$
  - $m \in \Gamma_X \iff X = A$

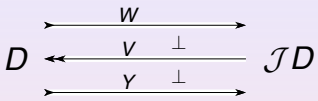
Domain of a protocol

$$D^{\mathcal{Q}} = \{a \in \wp\mathbb{T} \mid \forall \Theta \underset{fin}{\subseteq} a.\ \Theta \in \mathrm{Con}^{\mathcal{Q}} \wedge \Theta \vdash \Gamma \Rightarrow \Gamma \subseteq a\}$$

Order ideals

$$\mathcal{J}D = \left\{J \in D \mid \forall a \subseteq b \in J \Rightarrow a \in J \wedge \forall ab \in J\ \exists c \in J.\ a, b \subseteq c\right\}$$

Continuity = left adjoint to $\bigsqcup$



$$\begin{aligned} Y(a) &= \{x \sqsubseteq a\} \\ V(J) &= \bigsqcup J \\ W(a) &= \{x \ll a\} \end{aligned}$$

Continuity = left adjoint to $\bigsqcup$

Intuition

- $W(a) = \{x \ll a\}$ are the *key elements* of a
  - if $\bigsqcup J \sqsupseteq a$ is a "computation" of a
  - then $k \ll a$ means $k \in J$ for every such computation.
- $VW(a) = a$ means that a can be computed from its key elements.

Example: Encryption protocol

- $W(m) = \{m\}$

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- $\mathrm{Con}^{\mathcal{Q}}$ contains some sets that never occur
  - they cover the honesty assumptions

Example: Encryption protocol

- $W(m) = \{m\}$:(
  - although m is never sent in the clear
  - and no principal knows it without k
- $\mathrm{Con}^{\mathcal{Q}}$ contains some sets that never occur
  - they cover the honesty assumptions
- culprit: $\forall a \subseteq b \in J \Rightarrow a \in J$
  - every derivable term is derivable on its own

Honesty system of a protocol

Derivability

$$\Gamma \vdash \Theta \iff \forall t \in \Theta\ \exists \varphi \in \Sigma^*\ \exists \vec{g} \subseteq \Gamma.\ \varphi(g) \stackrel{E}{=} t$$

Honest sets

$$\begin{aligned} \Gamma^{\mathcal{Q}}_A &= A\text{'s environment in } \mathcal{Q} \\ \mathrm{Hon}^{\mathcal{Q}}_A &= \{\Theta \in \wp_{fin}\mathbb{T} \mid \Gamma^{\mathcal{Q}}_A \subseteq \Theta \wedge \Gamma^{\mathcal{Q}}_A \vdash \Theta\} \\ \mathrm{Hon}^{\mathcal{Q}} &= \bigcup_{A \in \mathcal{W}} \mathrm{Hon}^{\mathcal{Q}}_A \end{aligned}$$

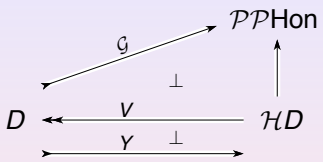
Domain of a protocol

$$D^{\mathcal{Q}} = \{a \in \wp\mathbb{T} \mid \forall \Xi \underset{fin}{\subseteq} a\ \exists \Theta \in \mathrm{Hon}^{\mathcal{Q}}.\ \Xi \subseteq \Theta \wedge \Xi \vdash \Gamma \Rightarrow \Gamma \subseteq a\}$$

Honest ideals

$$\mathcal{H}D = \Big\{ H \in D \;\Big|\; \forall a \subseteq b \in H \quad (\exists \Theta \in \mathrm{Hon}.\ \Theta \subseteq a) \Rightarrow a \in H \quad \wedge \quad \forall ab \in H\ \exists c \in H.\ a, b \subseteq c \Big\}$$

Guards = left multi-adjoint to $\bigsqcup$



$$a \sqsubseteq V(H) \iff \exists G \in \mathcal{G}(a).\ G \sqsubseteq H$$

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Example: Encryption protocol

- $\mathcal{G}(m) = \{\{k\}\}$:)

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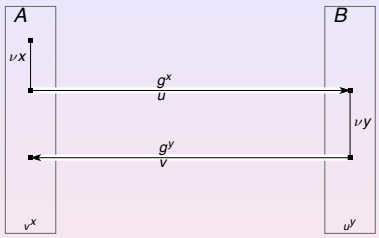
Example: Diffie-Hellman Key Agreement



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Example: Diffie-Hellman Key Agreement

- $W(g^{xy}) = \{g^{xy}\}$:(
- $\mathcal{G}(g^{xy}) = \{\{g^x, y\}, \{g^y, x\}\}$:)

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Algebraic theory of secrecy

(Meadows & DP)

$$\mathcal{G}\ \ \mathrm{guards}_\Upsilon \Theta \quad = \quad \forall t \in \Theta\ \forall \Xi \subseteq \Upsilon\ \exists \Gamma \in \mathcal{G}.\ \Xi \vdash t \Rightarrow \Xi \vdash \Gamma$$

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Algebraic theory of secrecy

(Meadows & DP)

$$\begin{aligned}
\mathsf{Have}(\Theta; G) &= \forall X \in G.\ \Gamma_X \vdash \Theta \\
\mathsf{Only}(\Theta; G) &= \forall X \in \mathcal{W}\ \forall t \in \Theta.\ \Gamma_X \vdash t \Rightarrow X \in G \\
\mathsf{Secr}(\Theta; G) &= \mathsf{Have}(\Theta; G) \wedge \mathsf{Only}(\Theta; G) \\
&= \forall X \in \mathcal{W}\ \forall t \in \Theta.\ \Gamma_X \vdash t \Leftrightarrow X \in G
\end{aligned}$$

Algebraic theory of secrecy

(Meadows & DP)

$$\frac{\mathsf{Have}(\Xi; G) \qquad \Xi \vdash_G \Theta}{\mathsf{Have}(\Theta; G)}$$

$$\frac{\mathsf{Only}(\Xi_i; G_i)\ |_{i=1}^n \qquad \{\Xi_i\}_{i=1}^n \text{ guards } \Theta}{\mathsf{Only}(\Theta; \bigcup_{i=1}^n G_i)}$$

$$\frac{\mathsf{Secr}(\Xi_i; G_i)\ |_{i=1}^n \quad \Xi_i \vdash_{G_i} \Theta\ |_{i=1}^n \quad \{\Xi_i\}_{i=1}^n \text{ guards } \Theta}{\mathsf{Secr}(\Theta; \bigcup_{i=1}^n G_i)}$$

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- unique decryption:

$$\mathsf{D}(\overline{k}, \mathsf{E}(k, m)) \quad = \quad m$$

- secrecy (IND-CCA):

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$$\left. b \in \mathbb{A}_3(c_0, m, m_0, m_1, c, c_1, \tilde{m}) \right) \ \leq \ \frac{1}{2}$$

for any probabilistic algorithm $\mathbb{A} = \langle \mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \rangle$

Outline

Idea

- assume that the algebra $\mathbb{T}$ is given with
  - an implementation

$$\begin{aligned} [\![-]\!] &: \mathcal{T} \to \mathcal{L} \\ \surd &: \mathcal{L} \to \mathcal{T} \end{aligned}$$

    such that $\surd[\![t]\!] = t$
  - monoid of feasible maps $\mathcal{F} \subseteq \mathcal{L}^{\mathcal{L}}$
  - frequency distribution

$$\mathsf{Prob} \ : \ \mathbb{T} \longrightarrow [0, 1]$$

Idea

- generalize
  - from algebraic derivability

$$\begin{array}{rcl} - \vdash - \ : \ D \times D & \longrightarrow & \{0, 1\} \\ \Gamma, \Theta & \longmapsto & \Gamma \vdash \Theta \end{array}$$

  - to Bayesian inference

$$\begin{array}{rcl} (- \vdash -) : D \times D & \longrightarrow & [0, 1] \\ \Gamma, \Theta & \longmapsto & \mathsf{Prob}(\Gamma \vdash \Theta) \end{array}$$

  - and guessing (cryptanalysis)

$$\begin{array}{rcl} [- \vdash -] : D \times D & \longrightarrow & [0, 1] \\ \Gamma, \Theta & \longmapsto & \bigvee_{\mathbb{A} \in \mathcal{F}} \mathsf{Prob}(\Gamma \vdash \Theta \in \mathbb{A}(\Gamma)) \end{array}$$

Hope

Develop

- a manageable formalization of guessing
- using information systems
- *enriched* over the ordered monoid $\Big([0,1],\ \cdot,\ 1,\ \leq\Big)$
- treat $(\Gamma \vdash \Theta)$ and $\big[\Gamma \vdash \Theta\big]$ as hom-objects in $[0,1]$
  - the states $\Gamma, \Theta \ldots$ can now be viewed as sets of equations *and inequalities*

Problem

The Bayesian inference and guessing are **not** transitive:

$$\begin{array}{rcl}(\Xi \vdash \Gamma)\cdot(\Gamma \vdash \Theta) & \not\leq & (\Xi \vdash \Theta)\\ \big[\Xi \vdash \Gamma\big]\cdot\big[\Gamma \vdash \Theta\big] & \not\leq & \big[\Xi \vdash \Theta\big]\end{array}$$

Problem

The Bayesian inference and guessing are **not** transitive:

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e.g., for

- $\Gamma = \emptyset$, thus $(\Xi \wedge \Gamma) = (\Xi)$
  - $(\Xi \vdash \Gamma) = \frac{(\Xi \wedge \Gamma)}{(\Xi)} = 1$
- $\Theta = \neg\Xi$
  - $(\Gamma \vdash \Theta) = (\emptyset \vdash \neg\Xi) = 1 - (\Xi)$
  - $(\Xi \vdash \Theta) = \frac{(\Xi \wedge \neg\Xi)}{(\Xi)} = 0$

Problem

. . . but they do satisfy

$$\begin{array}{rcl}(\Xi \vdash \Gamma)\cdot(\Xi, \Gamma \vdash \Theta) & = & (\Xi \vdash \Gamma, \Theta)\\ \big[\Xi \vdash \Gamma\big]\cdot\big[\Xi, \Gamma \vdash \Theta\big] & = & \big[\Xi \vdash \Gamma, \Theta\big]\end{array}$$

because

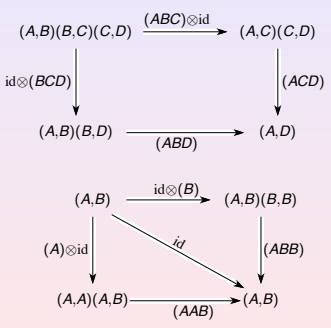
$$\frac{(\Xi\Gamma)}{(\Xi)}\cdot\frac{(\Xi\Gamma\Theta)}{(\Xi\Gamma)} = \frac{(\Xi\Gamma\Theta)}{(\Xi)}$$

$\mathbb{V}$-categories

- $(\mathbb{V}, \otimes, I)$
  - monoidal category, abbreviate $k \otimes \ell$ to $k\ell$
- $\mathbb{C} = \{A, B, \ldots\}$
  - class of objects
- $(A, B) \in \mathbb{V}$
  - hom-objects, for every $A, B \in \mathbb{C}$
- $(ABC) : (A, B) \otimes (B, C) \longrightarrow (A, C)$
  - composition, for every $A, B, C \in \mathbb{C}$
- $(A) : I \longrightarrow (A, A)$
  - identities, for every $A \in \mathbb{C}$

$\mathbb{V}$-categories

. . . satisfying



$\mathbb{V}$-categories

Examples

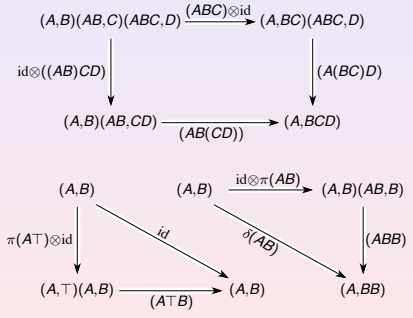
- $\mathbb{V} = \big(\{0,1\}, \wedge, 1\big)$ — preorders
- $\mathbb{V} = \big(\mathrm{Set}, \times, 1\big)$ — categories
- $\mathbb{V} = \big([0,\infty], +, 0\big)$ — metric spaces

$\mathbb{V}$-dependencies

- $(\mathbb{V}, \otimes, I)$
  - monoidal category, abbreviate $k \otimes \ell$ to $k\ell$
- $(\mathbb{C}, \cdot, \top)$
  - abelian monoid of objects, abbreviate $A{\cdot}B$ to AB
- $(A, B) \in \mathbb{V}$
  - hom-objects, for every $A, B \in \mathbb{C}$
- $(ABC) : (A, B) \otimes (AB, C) \longrightarrow (A, BC)$
  - composition, for every $A, B, C \in \mathbb{C}$
- $\pi(AB) : I \longrightarrow (AB, B)$ and $\delta(AB) : (A, B) \longrightarrow (A, BB)$
  - projections and diagonals, for every $A, B \in \mathbb{C}$

$\mathbb{V}$-dependencies

. . . satisfying



$\mathbb{V}$-dependencies

Examples

- $\mathbb{V} = \big(\{0,1\}, \wedge, 1\big)$ — semilattices

$$a \le b \wedge ab \le c \iff a \le bc$$

- $\mathbb{V} = \big(\mathrm{Set}, \times, 1\big)$ — dependent types (RCCCs)

$$\begin{array}{rcl} & (x : A & \triangleright \quad f(x) : B(x)) \\ \times & (x : A, y : B(x) & \triangleright \quad g(x,y) : C(x,y)) \\ \longrightarrow & (x : A & \triangleright \quad \langle f(x), g(x, f(x))\rangle : B(x) \times C(x, f(x)) \end{array}$$

- $\mathbb{V} = \big([0,1], \cdot, 1\big)$ — Bayesian nets

$$(A \vdash B){\cdot}(AB \vdash C) \quad = \quad (A \vdash BC)$$

Domain theory of Bayesian inference

Definition

Let $\mathbb{D}$ be a $[0,1]$-dependency. A *Bayesian ideal* is a map $\varphi : \mathbb{D} \longrightarrow [0,1]$ such that

$$(\Xi \vdash \Theta){\cdot}\varphi(\Xi\Theta) \quad \le \quad \varphi(\Xi)$$

We denote by $\mathcal{J}\mathbb{D}$ the dependency of Bayesian ideals, with the monoid and hom-object structure

$$\begin{array}{rcl} \varphi{\cdot}\psi(\Theta) & = & \varphi(\Theta){\cdot}\psi(\Theta) \\ (\varphi \vdash \psi) & = & \displaystyle\bigwedge_{\Theta \in \mathbb{D}} \left(\frac{\varphi(\Theta)}{\psi(\Theta)} \wedge \frac{\psi(\Theta)}{\varphi(\Theta)} \right) \end{array}$$

Domain theory of guessing

Definition

Let $\mathbb{D}$ be a $[0,1]$-dependency. A *guessing ideal* is an algorithm $\Phi : \mathbb{D} \longrightarrow [0,1]$ such that

$$\big[\Xi \vdash \Theta\big]{\cdot}\Phi(\Xi\Theta) \quad \le \quad \Phi(\Xi)$$

We denote by $\mathcal{H}\mathbb{D}$ the dependency of guessing ideals, with the monoid and hom-object structure

$$\begin{array}{rcl} \Phi{\cdot}\Psi(\Theta) & = & \Phi(\Theta){\cdot}\Psi(\Theta) \\ \big[\Phi \vdash \Psi\big] & = & \displaystyle\bigwedge_{\Theta \in \mathbb{D}} \left(\frac{\Phi(\Theta)}{\Psi(\Theta)} \wedge \frac{\Psi(\Theta)}{\Phi(\Theta)} \right) \end{array}$$

Ideals are Cauchy sequences

Theorem

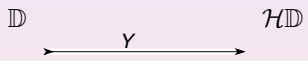
Bayesian (resp. guessing) ideals over a $[0,1]$*-dependency* $\mathbb{D}$ *correspond to the sequences of events (resp. guesses)* $\langle \Theta_i \rangle_{i=1}^{\infty}$ *such that*

$$\forall k \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall n > N\ \forall m \in \mathbb{N}.$$
$$(\Theta_1, \Theta_2, \ldots, \Theta_n \vdash \Theta_{n+m}) \ \geq\ e^{-\frac{1}{k}}$$

$$\exists N \in \mathcal{F}\ \forall k \in \mathbb{N}\ \forall n > N(k)\ \forall m \in \mathbb{N}.$$
$$[\Theta_1, \Theta_2, \ldots, \Theta_n \vdash \Theta_{n+m}] \ \geq\ e^{-\frac{1}{k}}$$

Guessing by adjoints

Fact.



$$Y(\Theta) \ =\ [- \vdash \Theta]$$

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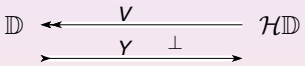
Guessing by adjoints

Proposition



$$Y(\Theta) \ =\ [- \vdash \Theta]$$
$$V(\Theta_i) \ =\ \lim_{i \to \infty} \Theta_i$$

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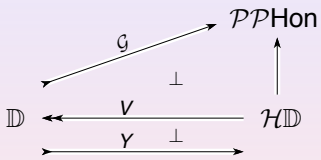

$$Y(\Theta) \ =\ [- \vdash \Theta]$$
$$V(\Theta_i) \ =\ \lim_{i \to \infty} \Theta_i$$
$$\mathcal{G} \text{ guards } \Theta \iff \forall \Xi.\ [\Xi \vdash \Theta] = \sum_{\Gamma \in \mathcal{G}} [\Xi \vdash \Gamma]\,[\Xi\Gamma \vdash \Theta]$$

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Way below IND-CCA

$$\begin{aligned} & & & [c_0] \\ & \cdot[c_0, m = \mathsf{D}(\overline{k}, c_0) & \vdash & \ m_0, m_1] \\ \cdot[c_0, m = \mathsf{D}(\overline{k}, c_0), m_0, m_1, c \in \mathsf{E}(k, m_b) & & \vdash & \ c_1 \neq c] \\ \cdot[c_0, m = \mathsf{D}(\overline{k}, c_0), m_0, m_1, c \in \mathsf{E}(k, m_b), & & & \\ c_1 \neq c, \widetilde{m} = \mathsf{D}(\overline{k}, c_1) & & \vdash & \ b] \ \leq\ [b] \end{aligned}$$

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Summary

- an algebraic theory of guessing can be presented as an algebraic theory of approximation
- a probabilistic theory of guessing can be presented by extending $\{0,1\}$-domains to $[0,1]$-domains