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On the structure of paradoxes

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Summary. Paradox is a logical phenomenon. Usually, it is produced in type theory, on a type Ω of "truth values". A formula ψ (i.e., a term of type Ω) is presented, such that $\psi \leftrightarrow \neg\psi$ (with negation as a term $\neg : \Omega \to \Omega$) – whereupon everything can be proved:

$$\dfrac{\dfrac{\psi\leftrightarrow\neg\psi \qquad \dfrac{\dfrac{\dfrac{\psi\leftrightarrow\neg\psi \quad {}^1[\psi]}{\neg\psi} \qquad {}^1[\psi]}{\bot}}{\neg\psi}\,1}{\psi} \qquad \neg\psi}{\bot}$$

In Sect. 1 we describe a general pattern which many constructions of the formula ψ follow: for example, the well known arguments of Cantor, Russell, and Gödel. The structure uncovered behind these paradoxes is generalized in Sect. 2. This allows us to show that Reynolds' [R] construction of a type $A \simeq \wp\wp A$ in polymorphic λ-calculus cannot be extended, as conjectured, to give a fixed point of *every* variable type derived from the exponentiation: for some (contravariant) types, such a fixed point causes a paradox.

Pursueing the idea that

$$\frac{\text{type theory}}{\text{categorical interpretation}} = \frac{\text{(propositional) logic}}{\text{Lindebaum algebra}}$$

the language of categories appears here as a natural medium for logical structures. It allows us to abstract from the specific predicates that appear in particular paradoxes, and to display the underlying constructions in "pure state". The essential role of cartesian closed categories in this context has been pointed out in [L]. The paradoxes studied here remain within the limits of the cartesian closed structure of types, as sketched in this Lawvere's seminal paper – and do not depend on any logical

operations on the type Ω. Our results can be translated in simply typed λ-calculus in a straightforward way (although some of them do become a bit messy).

1 Paradoxical structures

§ 1. Russell invented type theory in order to avoid paradoxes. On the other hand, the strongest paradoxes arise in a type-free setting. We will show that the germ of the most familiar paradoxes is contained in the *fixed point* (or *paradoxical*) *operator* of untyped λ-calculus:

$$Y(t) := \alpha(t) \cdot \alpha(t), \quad \text{where} \quad \alpha(t) := \lambda x.t \cdot (x \cdot x).$$

It satisfies the equation $Y(t) = t \cdot Y(t)$ for every term t, and produces a paradox by $\psi := Y(\neg)$.

This construction can not be repeated in an honest typed λ-calculus because it is in general impossible to apply a term on itself. However, the categorical interpretation of untyped λ-calculus shows that the fixed point operator can be constructed using many different notions of application: not only "the" application $f \cdot g$, but also the composition $f \circ g := \lambda x.f \cdot (g \cdot x)$ (also denoted fg), or even the substitution $f[x := g]$. (Other possibilities seem less interesting.)

§ 2. The categorical interpretation of untyped λ-calculus is well known (see [LSc]). It follows the same lines as the interpretation of typed λ-calculus (ibidem): the application and the abstraction are interpreted in the cartesian closed structure. The untyped case with the surjective pairing requires a category with one object – a monoid – which possesses all the cartesian closed structure, except, of course, the terminal object.

Definition. A *C-monoid* is a monoid $\mathbb{M}$ with the following adjunctions:
a) $\Delta\_ \dashv \_\times\_ : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$.
b) $M \times \_ \dashv M \to \_ : \mathbb{M} \to \mathbb{M}$, where M denotes the only object of $\mathbb{M}$.

Notation. $\mathbb{M} \times \mathbb{M}$ is the product in the category of categories, while $M \times M$ is the product in $\mathbb{M}$. Of course, since M is the only object of $\mathbb{M}$, we have $M \times M = M \to M = M$. The operation $\langle \_,\_ \rangle$ is the pairing in the category of categories, while $\langle\!\langle \_,\_ \rangle\!\rangle$ is the pairing in $\mathbb{M}$. Δ denotes the functor $\langle \mathrm{id}_{\mathbb{M}}, \mathrm{id}_{\mathbb{M}} \rangle : \mathbb{M} \to \mathbb{M} \times \mathbb{M}$.

Remark. The data of the adjunctions (a) and (b) are natural transformations which only have one component each. (a) is given by the transformation

$$\begin{aligned}&\pi_0, \pi_1 : M \to M \text{ and the correspondence}\\ &\langle\!\langle \_,\_ \rangle\!\rangle : \mathbb{M} \times \mathbb{M}(\Delta M, \langle M, M \rangle) \to \mathbb{M}(M, M \times M), \text{ such that}\\ &\langle\!\langle \pi_0 r, \pi_1 r \rangle\!\rangle = r \text{ and } \pi_i \langle\!\langle p_0, p_1 \rangle\!\rangle = p_i, \text{ for } i \in \{0, 1\}.\end{aligned}$$

On the other hand, (b) is determined by the transformation

$$\begin{aligned}&\varepsilon : M \to M \text{ and the correspondence}\\ &(\_)^* : \mathbb{M}(M \times M, M) \to \mathbb{M}(M, M \to M), \text{ which satisfy}\\ &\varepsilon \langle\!\langle r^* \pi_0, \pi_1 \rangle\!\rangle = r \text{ and } (\varepsilon \langle\!\langle q \pi_0, \pi_1 \rangle\!\rangle)^* = q.\end{aligned}$$

Construction. The fixed point operator in C-monoids can be obtained by a direct translation of the λ-term Y, constructed in 1. In this translation, the application would be defined:

$$f \cdot g := \varepsilon \langle\!\langle f(g\pi_1)^*, \mathrm{id} \rangle\!\rangle.$$

On the other hand, in every C-monoid holds

$$f \circ g = \varepsilon \langle\!\langle \ulcorner f \urcorner, g \rangle\!\rangle, \quad \text{where} \quad \ulcorner f \urcorner := (f\pi_1)^*.$$

Using this, we can obtain a fixed point operator *for composition*:

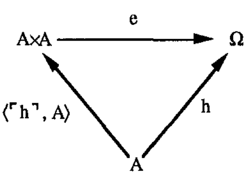
$$Y_t := \varepsilon\langle\!\langle \alpha_t, \alpha_t \rangle\!\rangle\,, \quad \text{where} \quad \alpha_t := \ulcorner t\varepsilon\langle\!\langle \mathrm{id}, \mathrm{id} \rangle\!\rangle \urcorner\,.$$

This construction now works in every cartesian closed category.

§ **3. Definition.** Let $\mathbb{C}$ (from now on) be a cartesian closed category, Ω an object in it. A *paradoxical structure* on Ω is a triple

$$\langle A \in |\mathbb{C}|,\ e \in \mathbb{C}(A \times A, \Omega),\ \ulcorner\_\urcorner : \mathbb{C}(A, \Omega) \to \mathbb{C}(\top, A)\rangle$$

(i.e., a "set", a "binary relation" and an "encoding of subsets"), such that the diagram



commutes for every h.

(Note that are we writing just $\ulcorner h \urcorner$ instead of $\ulcorner h \urcorner \circ \top_A : A \to \top \to A \times A$. With this convention is $h \circ g = e\langle \ulcorner h \urcorner, g\rangle$. We shall proceed with this abuse of notation whenever the confusion seems unlikely.)

Comment. The exponents were not used in this definition; nor will they be needed for the arguments which follow. However, they allow the paradoxical structures on Ω "at stage $X \in |\mathbb{C}|$" to be included in the above definition: they can be defined as the paradoxical structures on $X \to \Omega$. Without the exponents, one should have to consider less clean relations and encodings, namely

$$e \in \mathbb{C}(A \times A \times X, \Omega) \quad \text{and} \quad \ulcorner\_\urcorner : \mathbb{C}(A \times X, \Omega) \to \mathbb{C}(X, A)\,.$$

Theorem. *If there is a paradoxical structure on Ω, then every $t \in \mathbb{C}(\Omega, \Omega)$ has a fixed point $Y_t \in \mathbb{C}(\top, \Omega)$ – i.e., an arrow satisfying the equation $t \circ Y_t = Y_t$.*

Proof. Given a paradoxical structure $\langle A, e, \ulcorner\_\urcorner\rangle$ and an arrow $t \in \mathbb{C}(\Omega, \Omega)$, define

$$\alpha_t := \ulcorner t \circ e \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle \urcorner \quad \text{and} \quad Y_t := e \circ \langle \alpha_t, \alpha_t \rangle\,.$$

The following calculation shows that Y is the required fixed point:

$$\begin{aligned} Y_t &= e \circ \langle \alpha_t, \alpha_t \rangle = e \circ \langle \ulcorner t \circ e \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle \urcorner, \alpha_t \rangle = t \circ e \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle \circ \alpha_t \\ &= t \circ e \circ \langle \alpha_t, \alpha_t \rangle = t \circ Y_t\,. \quad \square \end{aligned}$$

Remark. The insight that the diagonal arguments like the one in the last proof can be formalized in cartesian closed categories is due to Lawvere [L]. It took some years and rediscoveries before the importance of Lawvere's logical approach to categories has been fully assessed.

§ **4. Examples. i. Russell's paradox.** Let $\mathbb{C}$ be the category of classes and (large) functions; so it contains an object $A :=$ Set, the class of all sets, and $\Omega := 2 := \{0, 1\}$. Define

$$e : \mathrm{Set} \times \mathrm{Set} \to 2 \quad \text{by} \quad e(P, Q) = 1 \quad \text{iff} \quad Q \in P\,.$$

Frege's unlimited comprehension just asserts the existence of an encoding

$$\ulcorner\_\urcorner : \mathbb{C}(\mathrm{Set}, 2) \to \mathbb{C}(1, \mathrm{Set})$$

which to every characteristic function $H : \mathrm{Set} \to 2$ (of a subclass of Set) assigns a set $\ulcorner H \urcorner$ such that

$$H(Q) = 1 \quad \text{iff} \quad Q \in \ulcorner H \urcorner\,.$$

Now take $t := \neg : 2 \to 2$, and consider the set

$$\alpha = \ulcorner \neg e\langle \mathrm{id}, \mathrm{id}\rangle \urcorner = \{x : x \notin x\}\,.$$

The truth value of $\alpha \in \alpha$ is

$$Y = e\langle \alpha, \alpha \rangle\,.$$

Calculating as in Theorem 3, we get

$$\neg Y = Y\,.$$

ii. Cantor's diagonalisation. Let $\mathbb{C}$ now be the category of sets and functions, and let $\Omega := 2$ be the additive group $\mathbb{Z}_2$ with two elements. Suppose that there is a paradoxical structure $\langle \mathbb{N}, e, \ulcorner\_\urcorner : 2^{\mathbb{N}} \to \mathbb{N}\rangle$. Thus just means that every sequence $h : \mathbb{N} \to 2$ must occur as a row in the matrix:

$$\begin{array}{llllll} e(1,1) & e(1,2) & e(1,3) & e(1,4) & \ldots & \\ e(2,1) & e(2,2) & e(2,3) & \ldots & & \\ e(3,1) & \ldots & & & & \\ \ldots & & & & & \\ e(\ulcorner h \urcorner, 1) & e(\ulcorner h \urcorner, 2) & e(\ulcorner h \urcorner, 3) & \ldots & e(\ulcorner h \urcorner, n)\,(= h(n)) \ldots & \\ \ldots & & & & & \end{array}$$

Now define

$$\begin{aligned} a(n) &:= e(n,n) + 1 \\ \alpha &:= \ulcorner a \urcorner \\ Y &:= e(\alpha,\alpha) = e(\ulcorner a \urcorner, \alpha) = a(\alpha,\alpha) = e(\alpha,\alpha) + 1 = Y + 1\,. \end{aligned}$$

iii. Gödel's incompleteness theorems can be interpreted in the cartesian closed category $\mathbb{C}$, freely generated by a single object $\mathbb{N}$, with the partial recursive functions as its endomorphisms. Let both $\Omega, A \in |\mathbb{C}|$ be the object $\mathbb{N}$. Choose an encoding $\ulcorner\_\urcorner$ of the formulas and terms of arithmetic, as it is usually done in recursion theory. From Gödel we know that the substitution can be encoded by a partial recursive function – so that in $\mathbb{C}$ there is an arrow

$$e := \mathrm{subst} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : \langle \ulcorner \varphi(x) \urcorner, n \rangle \mapsto \ulcorner \varphi[x := n] \urcorner\,,$$

where φ is a formula. (See, for instance, [S].) The triple $\langle \mathbb{N}, e, \ulcorner\_\urcorner \rangle$ is a paradoxical structure on $\mathbb{N}$, but this time *for the substitution*, not composition. Indeed, for every formula of arithmetic $\tau(x)$ there is a number $Y_\tau = \ulcorner \tau(Y_\tau) \urcorner$:

$$\begin{aligned} \alpha_\tau &:= \ulcorner \tau(e(x,x)) \urcorner \\ Y_\tau &:= r(\alpha_\tau, \alpha_\tau) = e(\ulcorner \tau(e(x,x)) \urcorner, \alpha_\tau) = \ulcorner \tau(e(\alpha_\tau, \alpha_\tau)) \urcorner = \ulcorner \tau(Y_\tau) \urcorner\,. \end{aligned}$$

Using this fact, Gödel composed the following variations on the theme of the liar paradox.

Impossibility of a definition of truth. Suppose that there is an arithmetical predicate True, such that for every formula φ holds

$$\vdash \varphi \leftrightarrow \mathrm{True}(\ulcorner \varphi \urcorner)\,.$$

Take $\tau := \neg\,\mathrm{True}$. Then

$$\psi := \tau(Y) = \neg\,\mathrm{True}(Y) = \neg\,\mathrm{True}(\ulcorner \tau(Y) \urcorner) = \neg\,\mathrm{True}(\ulcorner \psi \urcorner)\,.$$

First incompletenes theorem. The encoding $\ulcorner\_\urcorner$ of formulas of arithmetic can be extended on proofs. (This is standard material of recursion theory.) In this way, the provability ($\vdash$) of a formula can be expressed by an arithmetical predicate Pr, such that

$$\vdash \varphi \quad \textit{iff} \quad \vdash \mathrm{Pr}(\ulcorner \varphi \urcorner)$$

holds for every formula φ. Now consider the formulas $\tau: = \neg\mathrm{Pr}$ and $\nu: = \tau(Y)$. Since $\ulcorner\nu\urcorner = Y$ and $\nu = \neg\mathrm{Pr}(Y)$, we get

$$\begin{aligned} &\vdash \nu \Rightarrow \vdash \mathrm{Pr}(\ulcorner\nu\urcorner) \Rightarrow \vdash \mathrm{Pr}(Y) \Rightarrow \vdash \neg\neg\mathrm{Pr}(Y) \\ &\quad \Rightarrow \vdash \neg\nu\,, \\ &\text{hence } \nvdash \nu\,; \end{aligned}$$

on the other hand

$$\begin{aligned} &\vdash \neg\nu \Rightarrow \vdash \neg\mathrm{Pr}(\ulcorner\nu\urcorner) \Rightarrow \vdash \neg\mathrm{Pr}(Y) \\ &\quad \Rightarrow \vdash \nu\,, \\ &\text{thus } \nvdash \neg\nu\,. \end{aligned}$$

Hence, neither ν nor $\neg\nu$ are provable in arithmetic.

§ **5.** Other notions of substitution lead to just slightly different notions of self-reference.

Theorem. *If there is a paradoxical structure on Ω, then every polynomial $t(y) \in \mathbb{C}[y:\Omega](\top, \Omega)$ has a fixed point $Y_t \in \mathbb{C}(\top, \Omega)$ – in the sense that $t[y: = Y_t] = Y_t$. (Here is $\mathbb{C}[y:\Omega]$ the cartesian closed category freely generated by $\mathbb{C}$ and an arrow $y:\top \to \Omega$; see* [LSc]*).*

Proof. By the functional completeness, for each given polynomial $t(y)$ there is an arrow $a \in \mathbb{C}(A, \Omega)$, such that in $\mathbb{C}[x:A]$

$$t(e \circ \langle x, x\rangle) = a \circ x\,.$$

Denoting the paradoxical structure still by $\langle A, e, \ulcorner\_\urcorner\rangle$, we define

$$\begin{aligned} \alpha_t &: = \ulcorner a \urcorner\,, \\ Y_t &: = e \circ \langle \alpha_t, \alpha_t\rangle = a \circ \alpha_t = t(\varepsilon \circ \langle \alpha_t, \alpha_t\rangle) = t(Y_t)\,. \quad \Box \end{aligned}$$

§ **6.** In untyped λ-calculus the fixed point operator can be expressed as a term:

$$Y = \lambda t.(\lambda x.t \cdot (x \cdot x)) \cdot (\lambda x.t \cdot (x \cdot x))\,.$$

This is not always the case in categories. In fact, there are several degrees of paradoxicality.

Lemma. *Consider the mapping*

$$Y : \mathbb{C}(\Omega, \Omega) \longrightarrow \mathbb{C}(\top, \Omega) : t \mapsto Y_t\,,$$

defined in Theorem 3. It can be extended to a natural transformation

$$\psi : \mathbb{C}(\Omega \times \_, \Omega) \longrightarrow \mathbb{C}(\_, \Omega)\,,$$

with $\psi_\top = Y$ if and only if there is an arrow $y \in \mathbb{C}(\Omega \to \Omega, \Omega)$, such that

$$y \circ t^* = t \circ y \circ t^*$$

holds for every $t \in \mathbb{C}(\Omega, \Omega)$ (where $t^* \in \mathbb{C}(\top, \Omega \to \Omega)$ is the right transpose of t).

Proof. Yoneda. $\Box$

Proposition. *For a category $\mathbb{C}$ and an object $\Omega \in |\mathbb{C}|$, each of the following statements implies the next one.*

a) *$\mathbb{C}$ is a C-monoid.*

b) *There is an object $A \in |\mathbb{C}|$ and arrows $i \in \mathbb{C}(A \to \Omega, A)$ and $r \in \mathbb{C}(A, A \to \Omega)$, such that $r \circ i = \mathrm{id}$. (In other words, $A \to \Omega$ is a retract of A).*

c) *There is an arrow* fix $\in \mathbb{C}(\Omega \to \Omega, \Omega)$, *such that* $\mathrm{fix} \circ g^* = g \circ ((\mathrm{fix} \circ g^*) \times X)$ *holds for every* $g \in \mathbb{C}(\Omega \times X, \Omega)$.

d) *There is a paradoxical structure on* Ω.

Proof. (a) $\Rightarrow$ (b) is obvious. (c) $\Rightarrow$ (d) follows from the lemma.

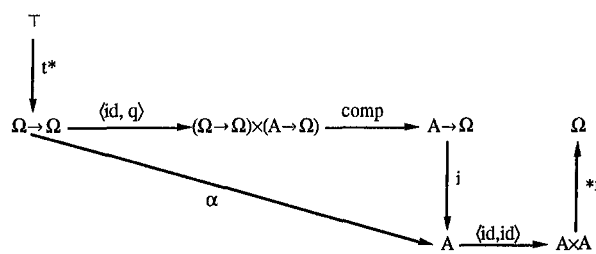
(b) $\Rightarrow$ (c): It is easy to see that $\langle A, {}^*r, i \circ (\_)^* \rangle$ is a paradoxical structure, where ${}^*r$ is the left transpose of r [i.e.${}^*r :=\varepsilon(r \times A)$]. Since the encoding $\ulcorner \_ \urcorner$ appears as arrow i, we can now perform the construction of the operator Y for this paradoxical structure internally, in order to get the arrow fix. Define

$$q := ({}^*r \circ \langle \mathrm{id}_A, \mathrm{id}_A \rangle)^* \circ \top : (\Omega \to \Omega) \to (A \to \Omega)$$

and

$$\mathrm{comp} := (\varepsilon_\Omega \circ ((\Omega \to \Omega) \times \varepsilon_A))^* : (\Omega \to \Omega) \times (A \to \Omega) \to A \to \Omega\,.$$

The construction of fixed point can be viewed on the following diagram:

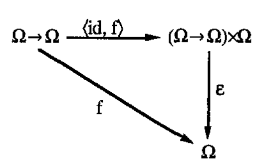


$\mathrm{fix} :={}^* r \circ \langle \alpha, \alpha \rangle$. □

Comment. So we have $\alpha_t = \alpha \circ t^*$. Note that the arrow fix induces a fixed point operator *at all stages* (in the sense of Remark 3).

§ **7. Proposition.** *There is a polynomial* $\varphi(z)$ *in the category* $\mathbb{C}[z : (\Omega \to \Omega) \to \Omega]$ *such that an arrow* $f \in \mathbb{C}(\Omega \to \Omega, \Omega)$ *satisfies the condition for* fix *in* 6(c) *if and only if* $\varphi(f^*) = f$.

Proof. It is easy to see that an arrow f is an internal fixed point operator at all stages [i.e. satisfies 6(c)] *iff* the following diagram commutes.



This means that $z := f^*$ is a fixed point of the polynomial

$$\varphi(z) := (\varepsilon_\Omega \circ \langle \mathrm{id}_{\Omega\to\Omega}, {}^*z \rangle)^* \quad \text{where} \quad {}^*z = \varepsilon_{\Omega\to\Omega} \circ \langle z \circ \top_{\Omega\to\Omega}, \mathrm{id}_{\Omega\to\Omega} \rangle\,. \quad \square$$

2 Generalized paradoxical structures

§ **8.** If an object Ω of a cartesian closed category is thought of as the set of truth values, the exponent $A \to \Omega$ is the powerset $\wp A$. So let us write $\wp A := A \to \Omega$.

The paradoxes which we considered so far, show – roughly speaking – that whenever $\wp A$ can be encoded in A, the logical structure of Ω gets destroyed, because all its endomorphisms – including negation – are then forced to have fixed points. How about $\wp\wp A$, $\wp\wp\wp A$, $\wp^n A$: could they by any chance be encoded in A? Can $\wp$ be an operation of finite order on some A?

Let us immediately say that a *positive* answer, provided by [R], has preceded these strange questions. Reynolds showed that for each inhabited type B in polymorphic λ-calculus with equality, there is a type A, such that $(A \to B) \to B \simeq A$. [Without equality, one could get a type A which contains $(A \to B) \to B$ as a retract.] If B is taken to be the type Ω of truth values, this construction gives a type A for which every element of $\wp\wp A$ is a "principal filter"! Of course, in set theory, this leads to a contradiction. Hence Reynolds' conclusion: "Polymorphism is not set-theoretic".

At the end of his paper, Reynolds suggested that his construction of a fixed point of the functor $((\_) \to B) \to B$ could be extended in polymorphic λ-calculus to every functor expressible using only the exponentaition $(\_) \to (\_)$. We disprove this for the contravariant functors $\wp^{2n+1}$. If any of them has a fixed point, then every endomorphism of Ω must have one. The argument is presented in the setting of cartesian closed categories – and can be translated in typed λ-calculus, without any use of polymorphism. A proof that Reynolds' conjecture holds for covariant functors has been provided in [RP].

§ **9.** *Notation.* As explained above, we shall write

$$\wp^0 X := X\,,$$
$$\wp^i X := \wp^{i-1} X \to \Omega\,.$$

Simplifying the subscripts of the evaluation arrows:

$$\varepsilon_i : \wp^{i+1} A \times \wp^i A \to \Omega$$

define the following families:

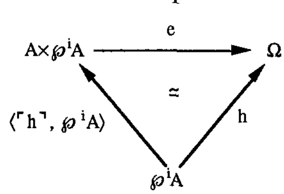
$$\varphi_k := (\varepsilon_{2k-2})^* : \wp^{2k-2} A \to \wp^{2k} A$$
$$\vartheta_1 := \varphi_1 : A \to \wp^2 A$$
$$\vartheta_i := \varphi_i \circ \vartheta_{i-1} : A \to \wp^{2i} A$$

– where the indices i and k go over natural numbers.

Definition. A *generalized paradoxical structure* of order i on an object Ω of a cartesian closed category $\mathbb{C}$ is a triple

$$\langle A \in |\mathbb{C}|,\ e \in \mathbb{C}(A \times \wp^i A, \Omega),\ \ulcorner\_\urcorner : \mathbb{C}(\wp^i A, \Omega) \to \mathbb{C}(\top, A)\rangle\,,$$

such that the following diagram commutes up to isomorphism



for every $h \in \mathbb{C}(\wp^i A, \Omega)$.

We call the paradoxical structures of order 0 *short*; the others are *long*.

Remark. "$e \circ \langle \ulcorner h \urcorner, \wp^i A\rangle \simeq h$" means that there are automorphisms c on Ω and d on $\wp^i A$ such that $e \circ \langle \ulcorner h \urcorner, \wp^i A\rangle = c \circ h \circ d$. The reason for this choice of definition is that some canonical isomorphisms ($X \times Y \simeq Y \times X, X \simeq \top \to X \dots$) occur in the fixed point theorem below, and they can in general not be removed. We get the fixed points only modulo canonical isomorphisms, and it seems natural to relax the other notions too. The isomorphisms have no repercussions for the logical questions which are relevant here. The difference between the short paradoxical structures defined here, and the paradoxical structures from Sect. 1 will therefore be neglected.

§ 10. Idea. The interpretation of Russell's paradox (Example 4.i.) suggests that a short paradoxical relation e could be understood as the *backwards* element relation $\ni$. In this way, the encoding $\ulcorner \_ \urcorner$ could be viewed as a realisation of the comprehension principle: for every $H \in \wp A$, we had

$$\forall x (H(x) \leftrightarrow \ulcorner H \urcorner \ni x)\,.$$

The long paradoxical relations are, however, easier to construct if we think of the paradoxical e as the relation $\in$. If the order is $i = 1$, then for every $H \in \wp\wp A$ holds

$$\forall X (H(X) \leftrightarrow \ulcorner H \urcorner \in X)\,.$$

The operation $\ulcorner \_ \urcorner$ now assigns a "limit" $\ulcorner H \urcorner$ to each "filter" $H \in \wp\wp A$.

For $i = 3$ and a "filter of filters" $H \in \wp\wp\wp\wp A$, the operation $\ulcorner \_ \urcorner$ returns a "limit of a limit":

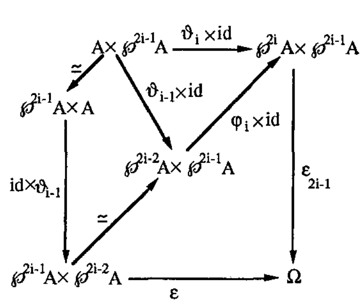
$$\begin{gathered}\exists H_1 \in \wp\wp A \forall X((H(X) \leftrightarrow H_1 \in X) \wedge \\ \forall X_1(H_1(X_1) \leftrightarrow \ulcorner H \urcorner \in X_1))\,;\end{gathered}$$

for $i = 5$, $H \in \wp^6 A$,

$$\begin{gathered}\exists H_2 \in \wp^4 A \forall X((H(X) \leftrightarrow H_2 \in X) \\ \wedge \exists H_1 \in \wp^2 A \forall X_2((H_2(X_2) \leftrightarrow H_1 \in X_2) \\ \wedge \forall X_1(H_1(X_1) \leftrightarrow \ulcorner H \urcorner \in X_1)))\,.\end{gathered}$$

This set-theoretical intuition leads us to the following constructions.

Lemma. $\varepsilon_{2i-1} \circ (\vartheta_i \times \wp^{2i-1} A) \simeq \varepsilon_{2i-2} \circ (\wp^{2i-1} A \times \vartheta_{i-1})$.
Proof.



Theorem. *If a generalized paradoxical structure of even order exists on Ω, then every $t \in \mathbb{C}(\Omega, \Omega)$ has a fixed point $Y \in \mathbb{C}(\top, \Omega)$.*

Proof. Let the order of paradoxical structure on $\Omega\langle A, e, \ulcorner\_\urcorner\rangle$ be $2k$. Define

$$\begin{aligned} a_0 &:= t \circ e \circ \langle \mathrm{id}_A, \vartheta_k \rangle : A \to \Omega\,, \\ a_i &:= \wp(a_{i-1}^*) : \wp^{2i} A \to \Omega \text{ for } 0 \leqq i \leqq k, \text{ and} \\ \alpha &:= \ulcorner a_k \urcorner\,, \\ Y &:= e\langle \alpha, \vartheta_k \circ \alpha \rangle\,. \end{aligned}$$

We shall prove that for every natural number i, $0 < i \leqq k$, there is an isomorphism

$$(\dagger) \qquad a_i \circ \vartheta_i \circ \alpha \simeq a_{i-1} \circ \vartheta_{i-1} \circ \alpha\,.$$

The required proof is then obtained by a descent along these isomorphisms:

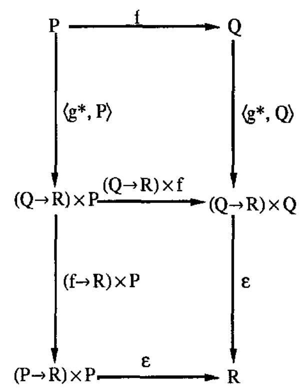
$$\begin{aligned} Y = e \circ \langle \alpha, \vartheta_k \circ \alpha \rangle &\simeq a_k \circ \vartheta_k \circ \alpha \simeq a_{k-1} \circ \vartheta_{k-1} \circ \alpha \simeq \cdots \simeq a_1 \circ \vartheta_1 \circ \alpha \simeq a_0 \circ \alpha \\ &\simeq t \circ e \circ \langle \alpha, \vartheta_k \circ \alpha \rangle = t \circ Y. \end{aligned}$$

A proof of (†):

$$\begin{aligned} a_i \circ \vartheta_i \circ \alpha = (a_{i-1}^* \to \Omega) \circ \vartheta_i \circ \alpha &\overset{(1)}{\simeq} {}^*(\vartheta_i \circ \alpha) \circ a_{i-1}^* \overset{(2)}{\simeq} \varepsilon_{2i-1} \circ \langle \vartheta_i \circ \alpha, a_{i-1}^* \rangle \\ &= \varepsilon_{2i-1} \circ (\vartheta_i \times \wp^{2i-1} A) \circ \langle \alpha, a_{i-1}^* \rangle \overset{(3)}{\simeq} \varepsilon_{2i-2} (\wp^{2i-1} A \times \vartheta_{i-1}) \circ \langle a_{i-1}^*, \alpha \rangle \\ &= \varepsilon_{2i-2} \circ \langle a_{i-1}^*, \vartheta_{i-1} \circ \alpha \rangle = a_{i-1} \circ \vartheta_{i-1} \circ \alpha\,. \end{aligned}$$

The marked steps are based on the following facts:

1) $(f \to R) \circ g^* = (gf)^*, f : P \to Q, g : Q \to R.$



Here we write g^* for $g^* \circ \top_p : P \to \top \to (Q \to R)$. Note that for $P := \top$ holds $(gf)^* \simeq gf$.

2) $\vartheta_i \circ \alpha = (\varepsilon_{2i-1} \circ (\vartheta_i \circ \alpha \times \wp^{2i-1} A))^*$.

3) Lemma. □

§ **11.** The preceding theorem allows variations analogous to those in §§ 5–7. Let us formulate only the statement which answers the question asked at the beginning of this section.

Corollary. *If there is an object A such that $\wp^{2k+1} A$ is a retract of A for some natural number k, then every endomorphism $t \in \mathbb{C}(\Omega, \Omega)$ has a fixed point.*

Proof. If $r \in \mathbb{C}(A, \wp^{2k+1}A)$ is a retraction, with $i \in \mathbb{C}(\wp^{2k+1}A, A)$, such that $ri = \mathrm{id}$, then

$$\langle A, {}^*r, i \circ (\_)^* \rangle$$

is a paradoxical structure of order $2k$ on Ω. □

§ **12. Comment.** In this paper, we saw an object Ω which tried to be the set of truth values, but another object A came along and disclosed that Ω was inconsistent. The paradoxical structure of A consisted of a relation $> := e$ and an encoding $\ulcorner \_ \urcorner$, which assigned to every predicate $\chi : A \to \Omega$ an element $\ulcorner \chi \urcorner$ of A such that

$$\chi(x) \leftrightarrow \ulcorner \chi \urcorner > x \,.$$

We then defined a formula α by

$$\alpha(x) \leftrightarrow \neg x > x$$

and obtained a paradoxical formula:

$$\psi : = \ulcorner \alpha \urcorner > \ulcorner \alpha \urcorner \leftrightarrow \alpha(\ulcorner \alpha \urcorner) \leftrightarrow \neg \ulcorner \alpha \urcorner > \ulcorner \alpha \urcorner = \neg \psi \,.$$

This scheme is suitable for a simple categorical interpretation, because it requires so little logic: one only has to decide to call an arrow to Ω formula, to call an endomorphism of Ω negation, and to interprete $\leftrightarrow$ as the equation (in Sect. 1) or an isomorphism (in Sect. 2). If, however, some more logic is provided, the concept of paradoxical structure can be modified in various ways. For instance, the main requirement on the encoding may be weakened to

$$\chi(x) \to \ulcorner \chi \urcorner > x \,.$$

In other words, χ need not be *conprehended*, but may be just *bounded* by $\ulcorner \chi \urcorner$. This is, of course, not very paradoxical – but it does lead to contradiction if there is a particular predicate ω which satisfies

$$\omega(x) \to \neg x > x \quad \text{and} \quad \omega(\ulcorner \omega \urcorner)$$

(i.e., its extension is irreflexive and closed with respect to the relation $>$). If ω is a description of the class of well-founded sets, this last scheme conveys the well-known paradox of Burali-Forti and Girard (cf. [TvD], Ch. 11, 7.4). Although it does not involve fixed points, the construction is clearly related with the paradoxical structures which we studied here.

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