A Treatise on

Quantum Clifford Algebras

Habilitationsschrift

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ABSTRACT: Quantum Clifford Algebras (QCA), i.e. Clifford Hopf gebras based on bilinear forms of arbitrary symmetry, are treated in a broad sense. Five alternative constructions of QCAs are exhibited. Grade free Hopf gebraic product formulas are derived for meet and join of Graßmann-Cayley algebras including co-meet and co-join for Graßmann-Cayley co-gebras which are very efficient and may be used in Robotics, left and right contractions, left and right co-contractions, Clifford and co-Clifford products, etc. The Chevalley deformation, using a Clifford map, arises as a special case. We discuss Hopf *algebra* versus Hopf *gebra*, the latter emerging naturally from a bi-convolution. Antipode and crossing are consequences of the product and co-product structure tensors and not subjectable to a choice. A frequently used Kuperberg lemma is revisited necessitating the definition of non-local products and interacting Hopf gebras which are generically non-perturbative. A 'spinorial' generalization of the antipode is given. The non-existence of non-trivial integrals in low-dimensional Clifford co-gebras is shown. Generalized cliffordization is discussed which is based on non-exponentially generated bilinear forms in general resulting in non unital, non-associative products. Reasonable assumptions lead to bilinear forms based on 2-cocycles. Cliffordization is used to derive time- and normal-ordered generating functionals for the Schwinger-Dyson hierarchies of non-linear spinor field theory and spinor electrodynamics. The relation between the vacuum structure, the operator ordering, and the Hopf gebraic counit is discussed. QCAs are proposed as the natural language for (fermionic) quantum field theory.

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“Al-gebra and Co-gebra
are brother and sister“

Zbigniew Oziewicz

Seht Ihr den Mond dort stehen
er ist nur halb zu sehen
und ist doch rund und schön
so sind gar manche Sachen
die wir getrost belachen
weil unsre Augen sie nicht sehn.

Matthias Claudius

Preface

This ‘Habilitationsschrift’ is the second incarnation of itself – and still in a *status nascendi.* The original text was planned to contain Clifford *algebras* of an arbitrary bilinear form, now called Quantum Clifford Algebras (QCA) and their beautiful application to quantum field theory (QFT). However, while proceeding this way, a major change in paradigm took place after the 5th Clifford conference held in Ixtapa 1999. As a consequence the first incarnation of this work faded away without reaching a properly typeset form, already in late 2000.

What had happened? During the 5th Clifford conference at Ixtapa a special session dedicated to Gian-Carlo Rota, who was assumed to attend the conference but died in Spring 1999, took place. Among other impressive retrospectives delivered during this occasion about Rota and his work, Zbigniew Oziewicz explained the Rota-Stein cliffordization process and coined the term ‘Rota-sausage’ for the corresponding tangle – for obvious reason as you will see in the main text. This approach to the Clifford product turned out to be superior to all other previously achieved approaches in elegance, efficiency, naturalness and beauty – for a discussion of ‘beautiness’ in mathematics, see [116], Chap. X, ‘*The Phenomenology of Mathematical Beauty*’. So I had decided to revise the whole writing. During 2000, beside being very busy with editing [4], it turned out, that not only a rewriting was necessary, but that taking a new starting point changes the whole tale!

A major help in entering the Hopf gebra business for Graßmann and Clifford algebras and cliffordization was the CLIFFORD package [2] developed by Rafał Abłamowicz. During a collaboration with him which took place in Konstanz in Summer 1999, major problems had been solved which led to the formation of the BIGEBRA package [3] in December 1999. The package proved to be calculationable stable and useful for the first time in Autumn 2000 during a joint work with Zbigniew Oziewicz, where many involved computations were successfully performed. The requirements of this lengthy computations completed the BIGEBRA package more or less. Its final form was produced jointly with Rafał Abłamowicz in Cookeville, September 2001.

The possibility of automated calculations and the knowledge of functional quantum field theory [128, 17] allowed to produce a first important result. The relation between time- and normal-ordered operator products and correlation functions was revealed to be a special kind of cliffordization which introduces an antisymmetric (symmetric for bosons) part in the bilinear form of the Clifford product [56]. For short, QCAs deal with time-ordered monomials while regular Clifford algebras of a symmetric bilinear form deal with normal-ordered monomials.

It seemed to be an easy task to translate with benefits all of the work described in [129, 48, 60, 50, 54, 55] into the hopfish framework. But examining Ref. [55] it showed up that the standard literature on Hopf algebras is set up in a too narrow manner so that some concepts had to be generalized first.

Much worse, Oziewicz showed that given an invertible scalar product B the Clifford bi-convolution $\mathcal{C}\ell(B, B^{-1})$, where the Clifford co-product depends on the co-scalar product B^{-1}, has *no antipode* and is therefore not a Hopf algebra at all. But the antipode played *the central role* in Connes-Kreimer renormalization theory [82, 33, 34, 35]. Furthermore the topological meaning and the group-like structure are tied to Hopf algebras, not to convolution semigroups. This motivated Oziewicz to introduce a second *independent* bilinear form, the co-scalar product C in the Clifford bi-convolution $\mathcal{C}\ell(B, C)$, $C \neq B^{-1}$ which is antipodal and therefore Hopf. A different solution was obtained jointly in [59].

Meanwhile QCAs made their way into differential geometry and showed up to be useful in Einstein-Cartan-Kähler theory with teleparallel connections developed by J. Vargas, see [131] and references therein. It was clear for some time that also differential forms, the Cauchy-Riemann differential equations and cohomology have to be revisited in this formalism. This belongs not to our main theme and will be published elsewhere [58].

Another source supplied ideas – geometry and robotics! – the geometry of a Graßmann-Cayley algebra, i.e. projective geometry is by the way the first *application* of Graßmann's work by himself [64]. Nowadays these topics can be considered in their relation to Graßmann Hopf gebras. The crucial 'regressive product' of Graßmann can easily be defined, again following Rota et al. [43, 117, 83, 11], by Hopf algebra methods. A different route also following Graßmann's first attempt is discussed in Browne [26]. Rota et al., however, used a Peano space, a pair of a linear space V and a volume to come up with invariant theoretic methods. It turns out, and is in fact implemented in BIGEBRA this way [6, 7], that meet and join operations of projective geometry are encoded most efficiently and mathematically sound using Graßmann Hopf gebra. Graßmannians, flag manifolds which are important in string theory, M-theory, robotics and various other objects from algebraic geometry can be reached in this framework with great formal and computational ease.

It turned out to be extremely useful to have geometrical ideas at hand which can be transformed into the QF theoretical framework. As a general rule, it is true that sane geometric concepts translate into sane concepts of QFT. However a complete treatment of the geometric background would have brought us too far off the road. Examples of such geometries would be Möbius geometry, Laguerre geometry, projective and incidence geometries, Hijelmslev planes and groups etc. [71, 15, 9, 10, 140]. I decided to come up with the algebraic part of Peano space, Graßmann-Cayley algebra, meet and join to have them available for later usage. Nevertheless, it will be possible for the interested reader to figure out to a large extend which geometric operations are behind many QF theoretical operations.

In writing a treatise on QCAs, I assume that the reader is familiar with basic facts about Graßmann and Clifford algebras. Reasonable introductions can be found in various text books, e.g. [115, 112, 14, 18, 27, 40, 87]. A good source is also provided by the conference volumes of the five international Clifford conferences [32, 93, 19, 42, 5, 120]. Nevertheless, the terminology needed later on is provided in the text.

In this treatise we make to a large extend use of graphical calculi. These methods turn out to be efficient, inspiring and allow to memorize particular equations in an elegant way, e.g. the ‘Rota-sausage’ of cliffordization which is explained in the text. Complicated calculations can be turned into easy manipulations of graphs. This is one key point which is already well established, another issue is to explore the topological and other properties of the involved graphs. This would lead us to graph theory itself, combinatorial topology, but also to the exciting topic of matroid theory. However, we have avoided graph theory, topology and matroids in this work.

Mathematics provides several graphical calculi. We have decided to use three flavours of them. I: Kuperberg’s translation of tensor algebra using a self-created very intuitive method because we require some of his important results. Many current papers are based on a couple of lemmas proved in his writings. II. Commutative diagrams constitute a sort of *lingua franca* in mathematics. III. Tangle diagrams turn out to be dual to commutative diagrams in a particular sense. From a physicist’s point of view they constitute a much more natural way to display dynamical ‘processes’.

Of course, graphical calculi are present in physics too, especially in QFT and for the tensor or spinor algebra, e.g. [106] appendix. The well known Feynman graphs are a particular case of a successful graphical calculus in QFT. Connes-Kreimer renormalization attacks QFT via this route. Following Cayley, rooted trees are taken to encode the complexity of differentiation which leads via the Butcher B-series [28, 29] and a ‘decoration’ technique to the Zimmermann forest formulas of BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) renormalization in momentum space.

Our work makes contact to QFT on a different and very solid way not using the mathematically peculiar path integral, but functional differential equations of functional quantum field theory, a method developed by Stumpf and coll. [128, 17]. This approach takes its starting point in position space and proceeds by implementing an algebraic framework inspired by and closely related to C^*-algebraic methods without assuming positivity.

However, this method was not widely used in spite of reasonable and unique achievements, most likely due to its lengthy and cumbersome calculations. When I became aware of Clifford algebras in 1993, as promoted by D. Hestenes [68, 69] for some decades now, it turns out that this algebraic structure is a key step to compactify notation and calculations of functional QFT [47]. In the same time many *ad hoc* arguments have been turned into a mathematical sound formulation, see e.g. [47, 48, 60, 50]. But renormalization was still not in the game, mostly since in Stumpf's group in Tübingen the main interest was laid on non-linear spinor field theory which has to be regularized since it is non-renormalizable.

While I was finishing this treatise Christian Brouder came up in January 2002 with an idea how to employ cliffordization in renormalization theory. He used the same transition as was employed in [56] to pass from normal- to time-ordered operator products and correlation functions but implemented an additional bilinear form which introduces the renormalization parameters into the theory but remains in the framework of cliffordization. This is the last part of a puzzle which is needed to formulate all of the algebraic aspects of (perturbative) QFT entirely using the cliffordization technique and therefore in the framework of a Clifford Hopf gebra (Brouder's term is 'quantum field algebra', [22]). This event caused a prolongation by a chapter on generalized cliffordization in the mathematical part in favour of some QFT which was removed and has to be rewritten along entirely hopfish lines. It does not make any sense to go with the *algebra only* description any longer. As a consequence, the discussion of QFT under the topic 'QFT as Clifford Hopf gebra' will be a sort of second volume to this work. Nevertheless, we give a complete synopsis of QFT in terms of QCAs, i.e. in terms of Clifford Hopf gebras. Many results can, however, be found in a pre-Hopf status in our publications.

What is the content and what are the *main results*?

- The Peano space and the Graßmann-Cayley algebra, also called bracket algebra, are treated in its classical form as also in the Hopf algebraic context.

- The bracket of invariant theory is related to a Hopf gebraic integral.

- Five methods are exhibited to construct (quantum) Clifford algebras, showing the outstanding beautiness of the Hopf gebraic method of cliffordization.

- We give a detailed account on Quantum Clifford Algebras (QCA) based on an arbitrary bilinear form B having no particular symmetry.

- We compare Hopf *algebras* and Hopf *gebras*, the latter providing a much more plain development of the theory.

- Following Oziewicz, we present Hopf gebra theory. The crossing and the antipode are exhibited as dependent structures which have to be calculated from structure tensors of the product and co-product of a bi-convolution and cannot be subjected to a choice.

- We use Hopf algebraic methods to derive the basic formulas of Clifford algebra theory (classical and QCA). One of them will be called *Pieri-formula of Clifford algebra*.
- We discuss the Rota-Stein cliffordization and co-cliffordization, which will be called, stressing an analogy, the *Littlewood-Richardson rule of Clifford algebra*.
- We derive *grade free* and very efficient product formulas for almost all products of Clifford and Graßmann-Cayley algebras, e.g. Clifford product, Clifford co-product (time- and normal-ordered operator products and correlation functions based on dotted and undotted exterior wedge products), meet and join products, co-meet and co-join, left and right contraction by arbitrary elements, left and right co-contractions, etc.
- We introduce non-interacting and interacting Hopf gebras which cures a drawback in an important lemma of Kuperberg which is frequently used in the theory of integrable systems, knots and even QFT as proposed by Witten. Their setting turns thereby out to be close to free theories.
- We show in low dimensional examples that no non-trivial integrals do exist in Clifford co-gebras and conjecture this to be generally true.
- A 'spinorial' antipode, a convolutive unipotent, is given which symmetrizes the Kuperberg ladder.
- We extend cliffordization to bilinear forms $\mathcal{BF}$ which are *not* derivable from the exponentiation of a bilinear form on the generating space B.
- We discuss generalized cliffordization based on non-exponentially generated bilinear forms. Assertions on the derived product show that exponentially generated bilinear forms are related to 2-cocycles.
- An overview is presented on functional QFT. Generating functionals are derived for time- and normal-ordered non-linear spinor field theory and spinor electrodynamics.
- A detailed account on the role of the counit as a 'vacuum' state is described. Two models with $U(1)$ and $U(2)$ symmetry are taken as examples.
- It is shown how the quantization enters the cliffordization. Furthermore we explain in which way the vacuum is determined by the propagator of the theory.
- Quantum Clifford algebras are proposed as the algebras of QFT.

What is *not to be found* in this treatise? It was not intended to develop Clifford algebra theory from scratch, but to concentrate on the 'quantum' part of this structure including the unavoidable hopfish methods. q-deformation, while possible and most likely natural in our framework is not explicitely addressed. However the reader should consult our results presented in Refs.

[51, 54, 5, 53] where this topic is addressed. A detailed explanation why 'quantum' has been used as prefix in QCA can be found in [57]. Geometry is reduced to algebra, which is a pity. A broader treatment, e.g. Clifford algebras over finite fields, higher geometries, incidence geometries, Hjielmslev planes etc. was not fitting coherently into this work and would have fatten it becoming thereby unhandsome. An algebro-synthetic approach to geometry would also constitute another volume which would be worth to be written. This is not a work in mathematics, especially not a sort of 'Bourbaki chapter' where a mathematical field is developed straightforward to its highest extend providing all relevant definitions and proving all important theorems. We had to concentrate on hot spots for lack of time and space and to come to a status where the method can be applied and prove its value. The symmetric group algebra and its deformation, the Hecke algebra, had to be postponed, as also a discussion of Young tableaux and their relation to Specht modules and Schubert varieties. And many more exciting topics . . .

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My gratitude goes to my wife Mechthild for her support, to my children simply for being there, and especially to my parents to whom this work is dedicated.

Konstanz, January 25, 2002
Bertfried Fauser

Wir armen Menschenkinder
sind eitel arme Sünder
und wissen garnicht viel
wir spinnen Luftgespinste
und suchen viele Künste
und kommen weiter von dem Ziel!

Matthias Claudius

Chapter 1

Peano Space and Graßmann-Cayley Algebra

In this section we will turn our attention to the various possibilities which arise if additional structures are added to a linear space ($\Bbbk$-module or $\Bbbk$-vector space). It will turn out that a second structure, such as a norm, a scalar product or a bracket lead to seemingly very different algebraic settings. To provide an overview, we review shortly normed spaces, Hilbert spaces, Weyl or symplectic spaces and concentrate on Peano or volume spaces which will guide us to projective geometry and the theory of determinants.

Let $\Bbbk$ be a ring or a field. The elements of $\Bbbk$ will be called scalars, following Hamilton. Let V be a linear space over $\Bbbk$ having an additively written group acting on it and a scalar multiplication. The elements of V are called vectors. Hamilton had a ‘vehend’ also and his vectors were subjected to a product and had thus an operative meaning, see e.g. [39]. We will also be interested mainly in the algebraic structure, but it is mathematical standard to disentangle the space underlying a ‘product’ from the product structure. Scalar multiplication introduces ‘weights’ on vectors sometimes also called ‘intensities’. As we will see later, the Graßmann-Cayley algebra does not really need scalars and is strictly speaking not an algebra in the common sense. We agree that an algebra A is a pair $A = (V, m)$ of a $\Bbbk$-linear space V and a product map $m : V \times V \to V$. Algebras are introduced more formally later. Products are mostly written in an *infix form*: $a\ m\ b \equiv m(a, b)$. Products are *defined* by Graßmann [64] as those mappings which respect distributivity w.r.t. addition, $a, b, c \in V$:

$$
\begin{aligned}
a\,m\,(b + c) &= a\,m\,b + a\,m\,c \\
(a + b)\,m\,c &= a\,m\,c + b\,m\,c
\end{aligned}
\tag{1-1}
$$

Hence the product is bilinear. Graßmann does not assume associativity, which allows to drop parentheses

$$
a\,m\,(b\,m\,c) = (a\,m\,b)\,m\,c. \tag{1-2}
$$

Usually the term algebra is used for ‘associative algebra’ while ‘non-associative algebra’ is used for the general case. We will be mostly interested in associative algebras.

1.1 Normed space – normed algebra

Given only a linear space we own very few rules to manipulate its elements. Usually one is interested in a reasonable extension, e.g. by a distance or length function acting on elements from V. In analytical applications it is very convenient to have a positive valued length function. A reasonable such structure is a norm $\|.\| : V \rightarrow \Bbbk$, a linear map, defined as follows

$$\begin{array}{rrl} o) & \|\alpha a\| = \alpha\, \|a\| & \alpha \in \Bbbk, \quad a \in V \\ i) & \|a\| = 0 & \text{if and only if } a \equiv 0 \\ ii) & \|a\| \geq 0 & \forall a \in V \quad \text{positivity} \\ iii) & \|a+b\| \leq \|a\| + \|b\| & \text{triangle relation.} \end{array} \tag{1-3}$$

As we will see later this setting is to narrow for our purpose. Since it is a strong condition it implies lots of structure. Given an algebra $A = (V, m)$ over the linear space V, we can consider a *normed algebra* if V is equipped additionally with a norm which fulfils

$$\|ab\| \leq \|a\|\,\|b\| \tag{1-4}$$

which is called *submultiplicativity*. Normed algebras provide a wealthy and well studied class of algebras [62].

However, one can prove that on a finite dimensional vector space all norms are equivalent. Hence we can deal with the prototype of a norm, the Euclidean length

$$\|x\|_2 := +\sqrt{\sum (x^i)^2} \tag{1-5}$$

where the $x^i \in \Bbbk$ are the coefficients of $x \in V$ w.r.t. an orthogonal generating set $\{e_i\}$ of V. We would need here the dual space V^* of linear forms on V for a proper description. From any norm we can derive an *inner product* by polarization. We assume here that $\Bbbk$ has only trivial involutive automorphisms, otherwise the polarization is more complicated

$$\begin{aligned} g(x,y) &: V \times V \rightarrow \Bbbk \\ g(x,y) &:= \|x-y\|. \end{aligned} \tag{1-6}$$

A 'distance' function also implies some kind of interpretation to the vectors as 'locations' in some space.

Since the major part of the work will deal with algebras over finite vector spaces or with formal power series of generating elements, i.e. without a suitable topology, thus dropping convergence problems, we are not interested in normed algebras. The major playground for such a structure is over infinitely generated linear spaces of countable or continuous dimension. Banach and C^*-algebras are e.g. of such a type. The later is distinguished by a C^*-condition which provides a unique norm, the C^*-norm. These algebras are widely used in non-relativistic QFT and statistical physics, e.g. in integrable models, BCS superconductivity etc., see [20, 21, 95].

1.2 Hilbert space, quadratic space – classical Clifford algebra

A slightly more general concept is to concentrate in the first place on an inner product. Let

$$
\begin{aligned}
< . \mid . > \; &: \; V \times V \rightarrow \Bbbk \\
< x \mid y > \; &= < y \mid x >
\end{aligned}
\tag{1-7}
$$

be a symmetric bilinear inner product. An inner product is called positive semi definite if

$$
< x \mid x > \; \geq \; 0 \tag{1-8}
$$

and positive definite if in the above equation equality holds if and only if $x \equiv 0$. The pair of a finite or infinite linear space V equipped with such a bilinear positive definite inner product $< . \mid . >$ is called a Hilbert space $\mathcal{H} = (V, < . \mid . >)$, if this space is closed in the natural topology induced by the inner product. Hilbert spaces play a prominent role in the theory of integral equations, where they have been introduced by Hilbert, and in quantum mechanics. The statistical interpretation of quantum mechanics is directly connected to positivity. Representation theory of operator algebras benefits from positivity too, e.g. the important GNS construction [95]. Of course one can add a multiplication to gain an algebra structure. This is a special case of a further generalization to quadratic spaces which we will consider now.

Let Q be a quadratic form on V defined as

$$
\begin{aligned}
Q \; &: \; V \rightarrow \Bbbk \\
Q(\alpha x) \; &= \; \alpha^2 Q(x) && \alpha \in \Bbbk, \quad x \in V \\
2\, B_p(x,y) \; &:= \; Q(x-y) - Q(x) - Q(y) && \text{where } B_p \text{ is bilinear.}
\end{aligned}
\tag{1-9}
$$

The symmetric bilinear form B_p is called polar bilinear form, the name stems from the pol-polar relation of projective geometry, where the locus of elements $x \in V$ satisfying $B_p(x,x) = 0$ is called *quadric*. However, one should be careful and introduce dual spaces for the 'polar elements', i.e. hyperplanes. It is clear that we have to assume that the characteristic of $\Bbbk$ is not equal to 2.

We can ask what kind of algebras arise from adding this structure to and algebra having a product m. Such a structure $A = (V, m, Q)$ would e.g. be an operator algebra where we have employed a non-canonical quantization, as e.g. the Gupta-Bleuler quantization of electrodynamics.

However, it is more convenient to ask if the quadratic form can *imply* a product on V. In this case the product map m is a consequence of the quadratic form Q itself. As we will see later, classical Clifford algebras are of this type. From its construction, based on a quadratic form Q having a symmetric polar bilinear form B_p, it is clear that we can expect Clifford algebras to be related to orthogonal groups. Classical Clifford algebras should thus be interpreted as a linearization of a quadratic form. It was Dirac who used exactly this approach to postulate his equation. Furthermore, we can learn from the polarization process that this type of algebra is related to anticommutation relations:

$$Q(x) = \sum_i x^i x^j e_i e_j$$
$$2\, B_p(x,y) = \sum_{i,j} x^i y^j (e_i e_j + e_j e_i) \tag{1-10}$$

which leads necessarily to

$$e_i\, e_j + e_j\, e_i = 2\, B_{p\,ij}. \tag{1-11}$$

Anticommutative such algebras are usually called (canonical) anticommutation algebras CAR and are related to fermions.

Classical Clifford algebras are naturally connected with the classical orthogonal groups and their double coverings, the **pin** and **spin** groups, [112, 113, 87].

Having generators $\{e_i\}$ linearly spanning V it is necessary to pass over to the linear space $W = \bigwedge V$ which is the linear span of all linearly and algebraically independent products of the generators. Algebraically independent are such products of the e_is which cannot be transformed into one another by using the (anti)commutation relations, which will be discussed later.

In the special case where the bilinear form on W, induced by this construction, is positive definite we deal with a Hilbert space. That is, Clifford algebras with positive (or negative) definite bilinear forms *on the whole space* W are in fact C^*-algebras too, however of a special flavour.

1.3 Weyl space – symplectic Clifford algebras (Weyl algebras)

While we have assumed symmetry in the previous section, it is equally reasonable and possible to consider antisymmetric bilinear forms

$$< . \mid . > : V \times V \to \Bbbk$$
$$< x \mid y > = - < y \mid x > . \tag{1-12}$$

A linear space equipped with an antisymmetric bilinear inner product will be called Weyl space. The antisymmetry implies directly that *all* vectors are null – or synonymously isotrop:

$$< x \mid x > = 0 \qquad\qquad \forall x \in V. \tag{1-13}$$

It is possible to define an algebra $A = (V, m, < . \mid . >)$, but once more we are interested in such products which are derived from the bilinear form. Using again the technique of polarization, one arrives this time at a (canonical) commutator relation algebra CCR

$$e_i\, e_j - e_j\, e_i = 2\, A_{ij}, \tag{1-14}$$

where $A_{ij} = -A_{ji}$. It should however be remarked, that this *symplectic Clifford algebras* are not related to classical groups in a such direct manner as the orthogonal Clifford algebras. The point is, that symplectic Clifford algebras do *not* integrate to a group action if built over a field [40, 18]. In fact one awaits nevertheless to deal with a sort of double cover of symplectic groups.

Such algebras are tied to bosons and occur frequently in quantum physics. Indeed, quantum physics was introduced for bosonic fields first and studied these much more complicated algebras in the first place.

In literature one finds also the name Weyl algebra for this type of structure.

There is an odd relation between the scalars and the symmetry of the generators – operators in quantum mechanics and quantum field theory. While for fermions the coefficients are commutative scalars forming a field and the generators are anticommutative we find in the case of bosons complicated scalars, at least a formal polynomial ring, or non-commutative coordinates. In combinatorics it is well known that such a vice-versa relation between coefficients and generators holds, see [66].

Also looking at combinatorial aspects, symplectic Clifford algebras are much more complicated. This stems from two facts. One is that one has to deal with multisets. The second is that the induced bilinear forms on the space W algebraically generated from V have in the antisymmetric case the structure of minors and determinants which are related to Pfaffians and obey decomposition, while in the symmetric case one ends up with permanents and Hafnians. The combinatorics of permanents is much more complicated.

It was already noted by Caianiello [30] that such structures are closely related to QFT calculations. We will however see below that his approach was not sufficient since he did not respect the symmetry of the operator product.

1.4 Peano space – Graßmann-Cayley algebras

In this section we recall the notion of a Peano space, as defined by Rota et al. [43, 11], because it provides the 'classical' part of QFT as a good starting point. Furthermore this notion is not well received. (In the older ref. [43] the term Cayley space was used). Peano space goes back to Giuseppe Peano's *Calcolo Geometrico* [105]. In this important work, Peano managed to surmount the difficulties of Graßmann's regressive product by setting up axioms in 3-dimensional space. In later works this goes under the name of the *Regel des doppelten Faktors* [rule of the (double) common factor], see the discussion in [26] where this is taken as an axiom to develop the regressive product. Graßmann himself changed the way how he introduced the regressive product from the first **A1** (**Ai** is common for the i-th 'lineale Ausdehnungslehre' [theory of extensions] from 1844 (A1) [64] and 1862 (A2) [63]) to the presentation in the **A2** . Our goal is to derive the wealth of products accompanying the Graßmann-Cayley algebra of meet and join, emerging from a 'bracket', which will later on be recast in Hopf algebraic terms. The bracket will show up as a Hopf algebraic integral of the exterior wedge products of its entries, see chapters below. The Graßmann-Cayley algebra is denoted bracket algebra in invariant theory.

1.4.1 The bracket

While we follow Rota et al. in their mathematical treatment, we separate explicitely from the comments about co-vectors and Hopf algebras in their writing in the above cited references. It is less known that also Rota changed his mind later. Unfortunately many scientists based their criticism of co-vectors or Hopf algebras on the above well received papers while the later change in the position of Rota was not appreciated, see [66, 119] and many other joint papers of Rota in the 90ies.

Let V be a linear space of finite dimension n. Let lower case x_i denote elements of V, which we will call also letters. We define a bracket as an alternating multilinear scalar valued function

$$\begin{aligned} [.,\ldots,.] \; &: \; V\times\ldots\times V \rightarrow \Bbbk \qquad\qquad n\text{-factors} \\ [x_1,\ldots,x_n] &= \text{sign}\,(p)[x_{p(1)},\ldots,x_{p(n)}] \\ [x_1,\ldots,\alpha x_r+\beta y_r,\ldots,x_n] \; &= \; \alpha[x_1,\ldots,x_r,\ldots,x_n]+\beta[x_1,\ldots,y_r,\ldots,x_n]. \end{aligned} \tag{1-15}$$

The sign is due to the permutation p on the arguments of the bracket. The pair $\mathcal{P}=(V,[.,\ldots,.])$ is called a Peano space.

Of course, this structure is much weaker as e.g. a normed space or an inner product space. It does not allow to introduce the concept of length, distance or angle. Therefore it is clear that a geometry based on this structure cannot be metric. However, the bracket can be addressed as a volume form. Volume measurements are used e.g. in the analysis of chaotic systems and strange attractors.

A *standard Peano space* is a Peano space over the linear space V of dimension n whose bracket has the additional property that for every vector $x\in V$ there exist vectors $x_2,\ldots,x_n$ such that

$$[x,x_2,\ldots,x_n]\neq 0. \tag{1-16}$$

In such a space the length of the bracket, i.e. the number of entries, equals the dimension of the space, and conversely. We will be concerned here with standard Peano spaces only.

The notion of a bracket is able to encode linear independence. Let x,y be elements of V they are linearly independent if and only if one is able to find $n-2$ vectors $x_3,...,x_n$ such that the bracket

$$[x,y,x_3,\ldots,x_n]\neq 0. \tag{1-17}$$

A *basis* of V is a set of n vectors which have a non-vanishing bracket. We call a basis *unimodular* or linearly ordered and normalized if for the ordered set $\{e_1,\ldots,e_n\}$, also called sequence in the following, we find the bracket

$$[e_1,\ldots,e_n]=1. \tag{1-18}$$

At this place we should note that an alternating linear form of rank n on a linear space of dimension n is uniquely defined up to a constant. This constant is however important and has to be removed for a fruitful usage, e.g. in projective geometry. This is done by introducing cross ratios. The group which maps two linearly ordered bases onto another is gl_n and sl_n for the mapping of unimodular bases.

1.4.2 The wedge product – join

To pass from a space to an algebra we need a product. For this reason we introduce equivalence classes of ordered sequences of vectors using the bracket. We call two such sequences equivalent

$$a_1, \ldots, a_k \cong b_1, \ldots, b_k \tag{1-19}$$

if for every choice of vectors $x_{k+1}, \ldots, x_n$ the following equation holds

$$[a_1, \ldots, a_k, x_{k+1}, \ldots, x_n] = [b_1, \ldots, b_k, x_{k+1}, \ldots, x_n]. \tag{1-20}$$

An equivalence class of this type will be called *extensor* or *decomposable antisymmetric tensor* or *decomposable k-vector*. The projection of the Cartesian product $\times$ (or the tensor product $\otimes$ if the $\Bbbk$-linear structure is considered) under this equivalence class is called *exterior wedge product of points* or simply wedge product if the context is clear. Alternatively we use the term *join* if geometrical applications are intended. In terms of formulas we find

$$a \wedge b := \{a\ ,\ b\} \quad \mathrm{mod}\ \cong \tag{1-21}$$

for the equivalence classes. The wedge product inherits antisymmetry from the alternating bracket and associativity, since the bracket was 'flat' (not using parentheses). Rota et al. write for the join the vee-product $\vee$ to stress the analogy to Boolean algebra, a connection which will become clear later. However, we will see that this identification is a matter of taste due to duality. For this reason we will stay with a wedge $\wedge$ for the 'exterior wedge product of points'. Furthermore we will see later in this work that it is convenient to deal with *different exterior products* and to specify them in a particular context. In the course of this work we even have occasion to use various exterior products at the same time which makes a distinction between them necessary. One finds 2^n linearly independent extensors. They span the linear space W which is denoted also as $\bigwedge V$. This space forms an algebra w.r.t. the wedge product, the *exterior algebra* or Graßmann algebra. The exterior algebra is a graded algebra in the sense that the module $W = \bigwedge V$ is graded, i.e. decomposable into a direct sum of subspaces of words of the same length and the product respects this direct sum decomposition:

$$\wedge\ :\ \bigwedge^r V \times \bigwedge^s V \rightarrow \bigwedge^{r+s} V. \tag{1-22}$$

The extensors of step n form a one dimensional subspace. Graßmann tried to identify this space also with the scalars which is not convenient [140]. Using an unimodular basis we can construct the element

$$E = e_1 \wedge \ldots \wedge e_n \tag{1-23}$$

which is called *integral*, see [130]. Physicists traditionally chose γ^5 for this element.

We allow extensors to be inserted into a bracket according to the following rule

$$\begin{aligned} A &= a_1, \ldots, a_r, \quad B = b_1, \ldots, b_s, \quad C = c_1, \ldots, c_t \\ [A, B, C] &= [a_1, \ldots, a_r, b_1, \ldots, b_s, c_1, \ldots, c_t] \\ n &= r + s + t. \end{aligned} \tag{1-24}$$

Since extensors are strictly speaking not generic elements, but representants of an equivalence class, it is clear that they are not unique. One can find quite obscure statements about this fact in literature, especially at those places where an attempt is made to visualise extensors as plane segments, even as circular or spherical objects etc. However an extensor A defines uniquely a linear subspace $\bar{A}$ of the space $\bigwedge V$ underlying the Graßmann algebra. The subspace $\bar{A}$ is called *support* of A.

A geometrical meaning of the join can be derived from the following. The wedge product of A and B is non-zero if and only if the supports of A and B fulfil $\bar{A} \cap \bar{B} = \emptyset$. In this case the support of $A \wedge B$ is the subspace $\bar{A} \cup \bar{B}$. Hence the join is the union of $\bar{A}$ and $\bar{B}$ if they do *not* intersect and otherwise zero – i.e. disjoint union. The join is an incidence relation.

If elements of the linear space V are called 'points', the join of two points is a 'line' and the join of three points is a 'plane' etc. One has, however, to be careful since our construction is till now characteristic free and such lines, planes, etc. may behave very oddly.

1.4.3 The vee-product – meet

The wedge product with multiplicators of step greater or equal than 1 raises the step of the multiplicand in any case. This is a quite asymmetric and geometrical unsatisfactory fact. It was already undertaken by Graßmann in the **A1** ('eingewandtes Produkt') to try to find a second product which lowers the step of the multiplicand extensor by the step of the multiplicator. Graßmann changed his mind and based his step lowering product in the **A2** on another construction. He also changed the name to 'regressives Produkt' [regressive product]. It might be noted at this place, that Graßmann denoted exterior products as 'combinatorisches Produkt' [combinatorial product] showing his knowledge about its link to this field.

Already in 1955 Alfred Lotze showed how the meet can be derived using combinatorial methods only [86]. Lotze considered this formula superior to the 'rule of the double factor' and called it 'Universalformel' [universal formula]. Lotze pointed clearly out that the method used by Graßmann in the **A2** needs a symmetric correlation, i.e. a transformation in projective geometry which introduces a quadric. However, Cayley and Klein showed that having a quadric is half the way done to pass over to metrical geometries. Mentioning this point seems to be important since in recent literature mostly the less general and less powerful method of the **A2** is employed. Zaddach, who was aware of Lotze's work [140], seemed to have missed the importance of this approach. The reader should also consult the articles of Zaddach p. 285, Hestenes p. 243, and Brini et al. p. 231 in [127] which exhibit tremendously different approaches.

We will shortly recall the second definition of the regressive product as given in the **A2** by Graßmann. First of all we have to define the 'Ergänzung' of an extensor A denoted by a vertical bar $|A$. Let A be an extensor, the Ergänzung $|A$ is defined using the bracket by

$$[A, |A] = 1. \tag{1-25}$$

From this equation it is clear that the 'Ergänzung' is a sort of orthogonal (!) complement or negation. But due to the fact that we consider disjoint unions of linear spaces, the present notion is more involved. We find for the supports of A and $|A$

$$\begin{aligned} \bar{A} \cap |\bar{A} &= \emptyset \\ \bar{A} \cup \overline{|A} &= \bar{E} \end{aligned} \tag{1-26}$$

where E is the integral. Furthermore one finds that the Ergänzung is involutive up to a possible sign which depends on the dimension n of V. Graßmann defined the regressive product, which we will call *meet* with Rota et al. and following geometrical tradition. The meet is derived from

$$|(A \vee B) := (|A) \wedge (|B) \tag{1-27}$$

which can be accompanied by a second formula

$$|(A \wedge B) = \pm(|A) \vee (|B) \tag{1-28}$$

where the sign once more depends on the dimension n. The vee-product $\vee$ is associative and anticommutative and thus another instance of an exterior product. The di-algebra (double algebra by Rota et al.) having two associative multiplications, sometimes accompanied with a duality map, is called *Graßmann-Cayley algebra*. The two above displayed formulas could be addressed as *de Morgan laws* of Graßmann-Cayley algebra. This implements a sort of logic on linear subspaces, a game which ships nowadays under the term quantum logic. It was Whitehead who emphasised this connection in his Universal Algebra.

The geometric meaning of the meet, which we denote by a vee-product $\vee$, is that of intersection. We give an example in $\dim V = 3$. Let $\{e_1, e_2, e_3\}$ be an unimodular basis, then we find

$$|e_1 = e_2 \wedge e_3 \qquad |e_2 = e_3 \wedge e_1 \qquad |e_3 = e_1 \wedge e_2. \tag{1-29}$$

If we calculate the meet of the following two 2-vectors $e_1 \wedge e_2$ and $e_2 \wedge e_3$ we come up with

$$\begin{aligned} & |((e_1 \wedge e_2) \vee (e_2 \wedge e_3)) = (e3 \wedge e1) \ = \ |e_2 \\ \Rightarrow \qquad & (e_1 \wedge e_2) \vee (e_2 \wedge e_3) = e_2 \end{aligned} \tag{1-30}$$

which is the *common factor* of both extensors. The calculation of the Ergänzung is one of the most time consuming operation in geometrical computations based on meet and join operations.

This renders the present definition of the meet as computational inefficient. Moreover, it is unsatisfactory that the meet is a ‘derived’ product and not directly given as the join or wedge.

The rule of the double [middle / common] factor reads as follows. Let A, B, C be extensors of step $a + b + c = n$ one assumes

$$(A \wedge C) \vee (B \wedge C) = (A \wedge B \wedge C) \vee C. \tag{1-31}$$

Using this relation one can express all regressive products in wedge products alone. Hence one is able to compute. However, also this mechanism renders the meet to be a derived and not a generic product.

Splits and shuffles: We will not follow Lotze’s presentation [86] of his ‘universal formula’ but for convenience the more recent presentation of Doubilet et al. [43]. First of all notation is much clearer there and secondly we will use their mechanism to derive a single wedge product of two factors, while Lotze computes a formula for the wedge product of r factors, motivating his ‘universal’ since it additionally does not need a symmetric correlation. Only thereafter the more general alternative laws could be derived which we have no occasion to consider in any depth here.

For convenience we *drop the wedge sign* for multiplication in the following. Note that the antisymmetry of elements allows to introduce a linear order in any sequence of vectors from V. We can e.g. use lexicographic ordering of letters or if we use indexed entities we can order by the value of the index. A word of $\bigwedge V$ (i.e. an extensor) is called *reduced* if it is ordered w.r.t. the chosen ordering. For instance

$$A = abcde \qquad B = b_1 b_2 \ldots b_r \qquad C = c_1 c_3 c_6 \tag{1-32}$$

are reduced words i.e. ordered, but

$$A = acebd \qquad B = b_4 b_2 b_3 b_1 b_5 \qquad C = c_6 c_1 c_3 \tag{1-33}$$

are not properly ordered w.r.t. the chosen ordering and need to be reordered. If one wants to come up with a basis for $\bigwedge V$ this is constituted by reduced words. Note that there are lots of orderings and it will be important to carefully distinguish them. In the following, we deal with reduced words (ordered basis extensors) only. A main problem in calculating the products is to expand the outcome into reduced basis elements. These are the straightening formulas of Rota et al. which could be called Littlewood-Richardson rule for Graßmann-Cayley algebra equivalently.

It would be a nice sidestep to study Young-tableaux, symmetric group representations and Specht modules, which we however resist to do in this work.

A *block* of an extensor is a subsequence (subword) extracted from the extensor (word). A $(\lambda_{i_1}, \ldots, \lambda_{i_k})$-*split* of an extensor A is the decomposition of the reduced word representing A into k blocks of length λ_{i_j} where $\sum \lambda_{i_s}$ = step A. E.g. $A = a \ldots bc \ldots de \ldots f$ is decomposed into $B_1 = (a \ldots b)$, $B_2 = (c \ldots d), \ldots$, $B_k = (e \ldots f)$. A *shuffle* of the $(\lambda_{i_1}, \ldots, \lambda_{i_k})$-split of

A is a permutation $p \in S_{\text{step}A+1}$ of A such that every block B_s remains to be reduced. In other words, the blocks B_s consist of ordered subsequences of letters from the word representing A.

The meet of k factors can be defined along the lines of Lotze using these shuffles and splits into k blocks. Rota et al. call these products bracket products. We will restrict ourselves to consider only (s,t)-splits into two blocks. Let $A = a_1 \ldots a_k$ and $B = b_1 \ldots b_s$ with $\dim V = n$ and $k + s \geq n$. We define the *meet* $\vee$ as

$$A \vee B = \sum_{\text{shuffles}} \text{sign}\,(p)[a_{p(1)}, \ldots, a_{p(n-s)}, b_1, \ldots, b_s]\; a_{p(n-s+1)} \wedge \ldots \wedge a_{p(k)} \tag{1-34}$$

where the permutations p range over all $(n-s, k-n+s)$-shuffles of $a_1 \ldots a_k$. Note the order of factors inside the bracket, which is given sometimes differently.

We introduce a *co-product* $\Delta \;:\; W \to W \otimes W$, which we will discuss later in detail, as the mapping of extensors A into a sum of tensor products of its $(n-s, k-n+s)$-shuffles of subsequences

$$\begin{aligned} \Delta(A) &= \sum_{\text{shuffles}} \text{sign}\,(p)\; a_{p(1)} \ldots a_{p(n-s)} \otimes a_{p(n-s+1)} \ldots a_{p(k)} \\ &= a_{(1)} \otimes a_{(2)} \end{aligned} \tag{1-35}$$

where we have introduced a shorthand known as Sweedler notation which implies the sum and the signs of the split as a sort of summation convention. Using this shorthand notation, the meet can be written as

$$A \vee B = [A_{(1)}, B]\, A_{(2)} \;=\; B_{(1)}[A, B_{(2)}]. \tag{1-36}$$

The second identity holds if and only if the particular order of factors is employed, otherwise a difference in sign may occur.

From this construction of the meet it is clear that no symmetric correlation is needed and consequently no Ergänzungs operator has to be employed. The BIGEBRA package [3] has both versions implemented as `meet` and `&v` products. There one can check the above identity on examples. Furthermore it turns out that the combinatorial implementation, which is ultimately based on Hopf algebra methods, is far more efficient than the above given and widely utilized method using the Ergänzung. Especially in robotics, where meet and join operations are frequently needed, this should speed up calculations dramatically [7]. For benchmarks see the online help-page of `meet` or `&v` from the BIGEBRA package.

1.4.4 Meet and join for hyperplanes and co-vectors

In projective geometry one observes a remarkable duality. If we consider a 3-dimensional projective space a correlation maps points into planes and planes into points. It is hence possible to consider planes as elementary objects and to construct lines and points by ‘joining’ planes. Projective duality shows that this geometry is equivalent to the geometry which considers points as basic objects and constructs lines and planes as joins of points. Recently projective duality was studied in terms of Clifford algebras [36, 37, 38]. Clifford algebras have been employed for projective geometry in e.g. [70]. However, the Clifford structure is essentially not needed, but was only introduced to compute the Ergänzung. Ziegler has described the history of classical mechanics in the 19th century [141] and showed there, that screw theory and projective methods have influenced the development of algebraic systems too. Graßmann considered (projective) geometry to be the first field to employ and exemplify his ‘new brach of mathematics’, see **A1**. Projective methods are widely used in image processing, camera calibration, robotics etc. [13].

However in these fields, engineers and applied mathematicians do not like co-vectors or tensor products, not to mention Hopf algebras. Rota et al. tried to cure the case by introducing co-vectors using the bracket, see [11], p. 122. They black-listed Bourbaki’s treatment [18] of co-vectors as follows: “Unfortunately, with the rise of functional analysis, another dogma was making headway at the time, namely, the distinction between a vector space V and its dual V^*, and the pairing of the two viewed as a bilinear form.” A few lines later, Hopf algebras are ruled out by stating that the “common presentation of both [interior and exterior products, BF] in the language of Hopf algebras, further obscures the basic fact that the exterior algebra is a bird of a different feather. ... If one insists in keeping interior products, one is sooner or later faced with the symmetry of exterior algebra as a Hopf algebra”. They develop a sort of co-vectors inside the bracket or Graßmann-Cayley algebra. We will see later, and Rota changed his mind also [66, 119], that this is not the proper way to deal with the subject. Indeed we have to reject even the term co-vector for this construction. We will call dual vectors introduced by the bracket as *reciprocal vectors*. It will turn out that reciprocal vectors need implicitly the Ergänzung and imply therefore the usage of a symmetric correlation. This introduces a distinguished quadric and spoils invariance under general projective transformations. Our criticism applies for the now frequently used homogenous models of hyperbolic spaces in terms of Clifford algebras [13].

If we identify vectors of the space V of dimension n with points, a hyperplane is represented by an extensor of step $n-1$. In other words, $n-1$ linearly independent points span a hyperplane. If hyperplanes are identified with reciprocal vectors, one can define an action of reciprocal vectors on vectors which yields a scaler. This motivated the misnaming of reciprocal vectors as co-vectors. We find using summation convention and an unimodular basis $\{e_i\}$ and the Ergänzung

$$
\begin{aligned}
&x \in V && x = x^i\, e_i \\
&u \in \bigwedge^{n-1} V && u = u^{i_1,\ldots,i_{n-1}} e_{i_1} \wedge \ldots \wedge e_{i_{n-1}} \\
& && u = u_k | e^k
\end{aligned}
\tag{1-37}
$$

where the Ergänzung yields the vector $e_{i_1} \wedge \ldots \wedge e_{i_{n-1}} \in \bigwedge^{n-1} V$, which we identify with the reciprocal vector e^k and the coefficients $u^{i_1,\ldots,i_{n-1}}$ are identified with u_k accordingly. Using the bracket one gets $[e_i, e^k] = \delta_i^k$. This reads for a vector x and a reciprocal vector u

$$u = u_k e^k\,. \tag{1-38}$$

We are able to use the bracket to write for the action $\bullet$ of a vector x on a reciprocal vector u

$$\begin{aligned} x \bullet u &= x^i u_k [e_i, e^k] \\ &= x^i u_k [e_i, |e_k] = x^i u_k \delta_i^k \\ &= x^i u_i \ \in \mathbb{k}. \end{aligned} \tag{1-39}$$

This mechanism can be generalized to an action of $\bigwedge^r V$ on $\bigwedge^{n-r} V$. The usage of the Ergänzung implying a quadric is pretty clear. This construction is used in [69] to derive 'co-vectors'. Hence all their formulas are not applicable in projective geometry which does not single out the Ergänzung or a symmetric correlation which implies a quadric.

However, we can follow Lotze, [86] note added in prove, to do the same construction but starting this time from the space of planes. Let $\vartheta \in V^*$ be a co-vector and $\{\vartheta^a\}$ be a set of canonical co-vectors dual to a basis $\{x_i\}$ of vectors spanning V, i.e. $\vartheta^a x_i = \delta_i^a$. One can form a Graßmann algebra on V^* along the same lines as given above by introducing a bracket on $\times^n V^*$. We denote the exterior product of this particular Graßmann algebra by vee $\vee$ that is the meet (join of hyperplanes). This reflects the fact that if it is allowed in a special case that co-vectors and reciprocal vectors are identified, their product is the meet. We can derive along the same lines as above a dual product called join. This join plays the same role to the above meet as the meet played beforehand to the join. It is denoted as 'join' (meet of hyperplanes) and uses the wedge $\wedge$ symbol, using splits and shuffles. It turns out, as our notation has anticipated, that this operation is the join of points, if points are identified as $n-1$-reciprocal vectors of co-vectors. We have experienced an instance of product co-product duality here, which will be a major topic in the later development of this treatise.

This consideration, which is exemplified to some detail in the online help-page of the `meet` and `&v` products of the BIGEBRA package [3], shows that it is a matter of choice which exterior product is used as meet and which as join by dualizing. This is the reason why we did not follow Rota et al. to use the vee-product $\vee$ for the join of points to make the analogy to Boolean algebra perfect.

However, we can learn an important thing. It is possible and may be necessary to implement an exterior algebra on the vector space V and the co-vector space V^* *independently*. This will give us a great freedom in the Hopf gebraic structure studied below. Moreover, it will turn out to be of utmost importance in QFT. Reordering and renormalization problems are hidden at this place. After our remarks it might not surprise that also classical differential geometry can make good use of such a general structure [131, 58].

Chapter 2

Basics on Clifford algebras

2.1 Algebras recalled

In this section we recall some definitions and facts from module and ring theory. In the sense we use the terms ‘algebra’ and ‘ring’, they are synonyms. We want to address the structure of the scalars as ring and the additive and distributive multiplicative structure on a module as algebra. The following statements about rings hold also for algebras.

From any book on module theory, e.g [134], one can take the following definitions:

Definiton 2.1. *A* ring *is a non-empty set R with two morphisms $+, \cdot\ :\ R \times R \to R$ fulfilling*

$$\begin{array}{lll} i) & (R,+) & \textit{is an abelian group, } 0 \textit{ its neutral element} \\ ii) & (R,\cdot) & \textit{is a semigroup} \\ iii) & (a+b)c = ab+bc & \forall\ a,b,c \in R \\ & a(b+c) = ab+ac & \end{array} \tag{2-1}$$

A ring R (same symbol for the underlying set and the ring) is called commutative, if $(R,\cdot)$ is commutative. If the multiplication map $\cdot$ enjoys associativity, the ring is called associative. We will assume associativity for rings.

An element $e \in R$ is called *left (right) unit* if $ea = a$ ($ae = a$) for all $a \in R$. A *unit* is a left and a right unit. A ring with unit is denoted *unital ring*.

The *opposite ring* R^{op} of R is defined to be the additive group $(R,+)$ with the opposite multiplication

$$a \circ^{op} b = b \cdot a. \tag{2-2}$$

A subgroup I of $(R,+)$ is called *left ideal* if $R \cdot I \subset I$ holds and *right ideal* if $I \cdot R \subset I$ holds. An *ideal* (also bilateral ideal) is at the same time a left and right ideal. If $(R,\cdot)$ is commutative then every ideal is a bilateral ideal. The intersection of left (right) ideals is again a left (right) ideal.

A *morphism of (unital) rings* is a mapping $f : (R, +, \cdot) \to (S, +, \circ)$ satisfying

$$\begin{aligned} f(a+b) &= f(a) + f(b) \\ f(a \cdot b) &= f(a) \circ f(b) \\ f(e_R) &= e_S \qquad\qquad \text{if } e_R, e_S \text{ do exist.} \end{aligned} \tag{2-3}$$

The kernel of a ring homomorphism $f : R \to S$ is an ideal

$$I_f = \ker f \; = \; \{a \in R \mid f(a) = 0\}. \tag{2-4}$$

The converse is true, every ideal is the kernel of an appropriate homomorphism. The *canonical projection* is given as $\pi_I : R \to R/I$ where R/I is the *residue class ring*. The ring structure in R/I is given as ($a, b \in R$)

$$\begin{aligned} (a+I) + (b+I) &= (a+b+I) \\ (a+I)(b+I) &= (ab+I). \end{aligned} \tag{2-5}$$

R/I is also called a *factor ring*.

Let A be a subset of R. An (left/right) ideal I_A is called *generated by* A if it is the smallest (left/right) ideal I_A with $A \subset I_A$. If A has finite cardinality we call I_A *finitely generated*. I_A is the intersection of all ideals which contain A.

The *direct sum* $A \oplus B$ of two ideals A, B is defined to be their Cartesian product $A \times B$ under the condition $A \cap B = \emptyset$. The ring R is called *decomposable* if it is a direct sum of (left/right) ideals $R = A \oplus B \oplus \ldots$, $A \cap B = \emptyset$, etc. In such rings every element r can be *uniquely* decomposed as

$$\begin{aligned} R \ni r &= a + b + \ldots \\ a \subset A, &\quad b \subset B, \ldots \end{aligned} \tag{2-6}$$

A ring is called (left/right) *indecomposable* if it cannot be written as a direct sum of (left/right) ideals. An analogous definition applies for ideals.

We define some special elements which will be needed later. An element a of the ring R is denoted as

- *left divisor of zero* if it exists a $b \neq 0$ such that $ab = 0$.
- *right divisor of zero* if it exists a $b \neq 0$ such that $ba = 0$.
- *divisor of zero* if it is a left and right divisor of zero.
- *idempotent* if $a^2 = a$.
- *nilpotent* (of order k) if $a^k = 0$.
- *unipotent* if R is unital and $a^2 = e$.

- *regular* if it exists an element $b \in R$ with $aba = a$.
- *left (right) invertible* if R is unital and it exists an element $b \in R$ such that $ab = e$ ($ba = e$).
- *invertible* if it is left and right invertible.
- *central* if for all $b \in R$ holds $ab - ba = 0$.

Two idempotents f_1, f_2 are called orthogonal if $f_1 f_2 = 0 = f_2 f_1$. An idempotent is called primitive if it cannot be written as the orthogonal sum of idempotents.

A subset A of R is called *left annulator* if $An^l(A) := \{b \in R \mid ba = 0, \quad \forall a \in R\}$, *right annulator* if $An^r(A) := \{b \in R \mid ab = 0, \quad \forall a \in R\}$, or *annulator* if $An^l(A) := An^l(A) \cap An^r(A)$.

Theorem 2.2 (Direct decomposition). *Let R be a ring, it holds*

1) *If the left ideal $I \subset R$ is generated by an idempotent $f \in R$, $I = Rf$, then R is decomposable into left ideals $R = A \oplus An^l(f)$.*
2) *If the ideal J is generated by a central idempotent f then R is decomposable into $R = J + An(f)$.*
3) *Let R be an unital ring. Every (left/right) ideal I which is a direct summand is generated by an idempotent element f. If I is an ideal then f is central. The decomposition is $R = Rf + An^l(F)$, where $An^l(f) = R(1 - f)$.*

Proof: see [134].

2.2 Tensor algebra, Graßmann algebra, Quadratic forms

Our starting point to construct Clifford algebras and later on Clifford Hopf gebras will be the Graßmann algebra. We have already used the language of an alphabet having letters which do form words to introduce this mathematical structure in the chapter on the Peano bracket. Hence we will introduce here the same structure by factoring out an ideal from tensor algebra. We will have occasion to use this technique later on.

Let $\Bbbk$ be an unital commutative ring and let V be a $\Bbbk$-linear space. The tensor algebra $T(V)$ is formed by the direct sum of tensor products of V

$$\begin{aligned} T(V) &= \Bbbk \,\oplus\, V \,\oplus\, (V \otimes V) \,\oplus \ldots \\ &= \oplus_r T^r(V) \;=\; \oplus_r \otimes^r V. \end{aligned} \tag{2-7}$$

We identify $\Bbbk$ with V^0 in a canonical way. The unit of $\Bbbk$ in $T(V)$ is denoted as Id. The injection $\eta \;:\; \Bbbk \to T(V)$ into the tensor algebra will be needed below and is called *unit map*, also denoted Id_V. The elements of the set $\{e_i\}$ of linearly independent elements which span V are called set of generators. The words obtained from these generators by concatenation yields a basis of $T(V)$. All elements of V are called letters, decomposable elements of $\otimes^r V$, i.e. $a_1 \otimes \ldots \otimes a_r \in T^r(V) \equiv \otimes^r V$. The number of factors is called *length* of the word or *rank* of the tensor. One can add words of the same length which will in general lead to an indecomposable tensor, but still a tensor of the same rank. Sums of words of same or arbitrary length might be called sentences. The tensors of a particular rank form a linear subspace of $T(V)$. Products of tensors are formed by concatenation of words,

$$(a_1 \otimes \ldots \otimes a_r) \otimes (b_1 \otimes \ldots \otimes b_s) = a_1 \otimes \ldots \otimes a_r \otimes b_1 \otimes \ldots \otimes b_s. \tag{2-8}$$

Concatenation is by definition associative. $T(V)$ is naturally graded by the length or rank of the tensors. i.e. products of r-tensors and s-tensors are $r+s$-tensors. For a precise definition of the tensor product look up any algebra book, [124, 125].

The Graßmann algebra is obtained by projecting the tensor product onto the antisymmetric wedge product $\pi(\otimes) \rightarrow \wedge$. In the case of the Graßmann algebra, we can either describe the equivalence class or deliver relations among some generators. If relations hold, not all words of the tensor algebra which can be formed by concatenation remain to be independent. The problem to identify two words w.r.t. given relations is called the *word problem*. It can in general not be solved, however, we will deal with solvable cases here. To be able to pick a representant from an equivalence class, we have to define *reduced words*. A reduced word is semi *ordered* in a certain sense. One has to use the relations to establish such a semi ordering, sometimes called *term ordering* in the theory of Gröbner bases.

We define the following ideal which identifies all but antisymmetric tensors

$$\mathcal{I}_\wedge = \{a \otimes x \otimes x \otimes b \mid a, b \in T(V), \quad x \in V\}. \tag{2-9}$$

The Graßmann algebra $\bigwedge V$ is the factor algebra of $T(V)$ where the elements of the above given ideal are identified to zero.

$$\begin{aligned} \bigwedge V &= \frac{T(V)}{\mathcal{I}_\wedge} \\ &= \pi_\wedge(T(V)) \end{aligned} \tag{2-10}$$

where $\pi_\wedge$ is the canonical projection from $T(V)$ onto $\bigwedge V$. From this construction it is easy to show by means of categorial methods that a Graßmann algebra over a space V is a *universal object* and is defined uniquely up to isomorphy.

The relations which are equivalent to the above factorization read

$$\begin{aligned} e_i \otimes e_i &= 0 \mod \mathcal{I}_\wedge \\ \pi_\wedge(e_i \otimes e_i) &= e_i \wedge e_i \; = \; 0 && \text{(2-11)} \\ \pi_\wedge(e_i \otimes e_j) &= e_i \wedge e_j \; = \; -e_j \wedge e_i. && \text{(2-12)} \end{aligned}$$

While the tensor algebra had essentially no calculational rules to manipulate words or sentences, beside multilinearity, one has to respect such relations after factorization. We can introduce reduced words by asserting that words of generators are ordered by ascending (descending) indices. A basis of $\bigwedge V$ is given as

$$GB = \{\mathrm{Id}; e_1, \ldots, e_n; e_1 \wedge e_2, \ldots e_{n-1} \wedge e_n; \ldots; e_1 \wedge \ldots \wedge e_n\} \tag{2-13}$$

where we have separated words of a different length by a semi-colon. Due to the relations we find for a finitely generated space V of dimension n a *finite number* of reduced words only. Their number is $\sum \binom{n}{r} = 2^n$. The space spanned by these generators will be called $W = \bigwedge V$. In analogy to the group theory [74] we can define a *presentation of an algebra* over V spanned by the set of generators X as follows:

$$\begin{aligned}\mathrm{Alg}(V) =&\ \langle X, \mathbf{R} \rangle \\ =&\ \{\, (\{e_i\}, \{\mathbf{R_i}\}) \mid V = \text{ span}\{e_i\}, \mathbf{R_i} \text{ relations}\,\}.\end{aligned} \tag{2-14}$$

We will freely pass from one picture to the other as it is convenient. The techniques from group presentations and terminology, e.g. word problem, generator, etc. can be applied to algebras by analogy. E.g. a free algebra is an algebra generated by a set X of generators e_i which span V having no relations at all. A free Lie algebra has of course relations which renders it to be a Lie algebra, but no further constraints among its 'Lie words'.

We had already occasion to define quadratic forms previously, so we recall here only the basis free definition

$$\begin{aligned}Q(\alpha\, x) &= \alpha^2\, Q(x) \\ 2\; B_p(x, y) &= Q(x - y) - Q(x) - Q(y) \qquad\qquad B_p \text{ bilinear.}\end{aligned} \tag{2-15}$$

As we pointed out, the addition of a quadratic form to a linear space yields a quadratic space. The main idea of a Clifford algebra is to form an algebra in a natural way from this building blocks. One can show that there is a functorial relation between quadratic spaces and associative unital algebras. This functor is injective and denoted as $\mathcal{C}\ell$. It is clear from this observation that the classification of Clifford algebras is essentially given by the classification of the quadratic forms used in their construction. If $\Bbbk$ is $\mathbb{R}$ or $\mathbb{C}$, this can be readily done by signature and dimension in the case of $\mathbb{R}$ or dimension only in the case of $\mathbb{C}$.

In the following sections we will provide some possible methods to establish this functorial relation. Each method has its advantages in certain circumstances, so none has to be abandoned, however, we will spend lots of efforts to provide a universal, computationally efficient, and sound approach to Clifford algebras, which will turn out to be Rota-Stein cliffordization. We will in the same time generalize the term Clifford algebra to *Quantum Clifford algebra* (QCA) if we consider algebras built from spaces having a bilinear form of arbitrary symmetry. It will turn out during our treatment of the subject that we will need necessarily the co-algebra and Hopf algebra structure which is hidden or implicit in the more basic approaches. Hopf techniques will be extremely helpful in applications, speeding up actual computations, e.g. of meet and join, used in robotics. The same holds true for Clifford products, [6, 7]. Cliffordization turns out to be a neat device to describe normal-, time-, and even renormalized time-ordered operator products and correlation functions in QFT.

2.3 Clifford algebras by generators and relations

The generator and relation method is the historical root of several algebraic systems. Hamilton's quaternion units $\mathbf{i},\mathbf{j},\mathbf{k}$ are still used in vector analysis, Graßmann used basis vectors e_i to generate his 'Hauptgebiet', our linear space V. A basis independent method was in general not available during these times, hence, also Clifford introduced and studied algebras in terms of generators and relations. The presentation of a Clifford algebra is as follows:

$$\begin{aligned}\mathcal{C}\ell(V,Q) &=< X, R > \\ &=< \{e_i\}, e_ie_j + e_je_i = 2\, g_{ij} > \end{aligned} \tag{2-16}$$

where the $e_i \in X$ span V and g_{ij} is the symmetric polar bilinear form which represents Q in the basis of the generators. These relations are usually called (anti)commutation relations. In physics only the commutation relations of the generators are usually given to define algebras, hence one writes

$$e_ie_j + e_je_i = 2\, g_{ij}. \tag{2-17}$$

Synonymous notations are $\mathcal{C}\ell(V,Q)$, $\mathcal{C}\ell(Q)$ if V is clear, $\mathcal{C}\ell_{p,q,r}$ if V is an $\mathbb{R}$-linear space of dimension $p+q+r$, while the quadratic form has p positive, q negative eigenvalues and a radical of dimension r, and $\mathcal{C}\ell_n$ if V is a $\mathbb{C}$-linear space of dimension n. The Clifford product is denoted by juxtaposition or if we want to make it explicite by a circle $\circ$, sometimes called circle product [119]. A natural basis for this algebra would be the Clifford basis, ordered by ascending indices

$$CB = \{\mathrm{Id}; e_1, \ldots, e_n; e_1e_2, \ldots e_{n-1}e_n; \ldots; e_1 \ldots e_n\} \tag{2-18}$$

which does not resort to the Graßmann exterior product. But most applications actually use a Graßmann basis. Such a basis is obtained by antisymmetrization of the Clifford basis elements, e.g.

$$e_i \wedge e_j = \frac{1}{2}(e_ie_j - e_je_i). \tag{2-19}$$

It was shown by Marcel Riesz [115] that a wedge product can be consistently developed in a Clifford algebra. This basis is isomorphic to a basis of a Graßmann algebra. Hence it is clear, that Clifford and Graßmann algebras have the same dimension. We will see below, that one can construct Clifford algebras as a subalgebra of the endomorphism algebra of an underlying Graßmann algebra.

The most remarkable changes between a Graßmann and a Clifford algebra are, that the latter has a richer representation theory. This stems from the fact that in a Graßmann algebra Id and 0 are the only idempotent elements. That is $\bigwedge V$ is an indecomposable algebra. One finds beside nilpotent ideals only trivial ideals. Clifford algebras have idempotent elements which generate various spinor representations. This fact follows directly from the quadratic form introduced in the Clifford algebra.

We had noted that the Graßmann basis GB spans a $\mathbb{Z}$-graded linear space. The exterior wedge product was graded too. Since the Clifford algebra can be described using a Graßmann basis, it seems to be possible to introduce a $\mathbb{Z}$-grading here also. However, a short calculation shows that the Clifford product does *not* respect this grading, but only a weaker filtration, see later chapters. Let u, v be extensors of step r and s one obtains

$$u \circ v \in \oplus_{n=|r-s|}^{r+s} \bigwedge^n V. \tag{2-20}$$

This is not an accident of the foreign basis, but remains to be true in a Clifford basis also. The terms of lower step emerge from the necessary commutation of some generators to the proper place in a reduced word. For instance

$$(e_1 e_2 e_5) \circ (e_3 e_4 e_6) = (e_1 e_2 e_3 e_4 e_5 e_6) + g_{35}(e_1 e_2 e_4 e_6) - g_{45}(e_1 e_2 e_3 e_6). \tag{2-21}$$

As a matter of fact a Clifford algebra is only Z_2-graded since even- and oddness of the length of words is preserved. The commutation relation contracts two generators for each commutation.

The usually defined *grade projection operators* $< \ldots >_r \,:\, \bigwedge V \rightarrow \bigwedge^r V$ are foreign to the concept of a Clifford algebra and belongs to the underlying Graßmann algebra. We will see later, that one is able to employ various $\mathbb{Z}$-gradings at the same time. It will be of great importance to keep track of the grading which is inherited from the Graßmann algebra. However, the mere *choice* of a set of generators $\{e_i\}$ induces a $\mathbb{Z}_n$-grading w.r.t. an underlying Graßmann algebra. The question if such representations are equivalent is known as *isomorphy problem* in the theory of group presentations [74]. In fact it is easy to find, e.g. using CLIFFORD [2], non grade preserving transformations of generators. This is well known from the group theory. E.g. the braid group on three strands has presentations

$$\begin{aligned} B_3 &=< \{x, y\}, xyx = yxy > \qquad \text{or} \\ B_3 &=< \{a, b\}, a^3 = b^2 > \end{aligned} \tag{2-22}$$

where one sets with $xy = a$ and $x = a^{-1}b$

$$y = x^{-1}a, \qquad\qquad ax = x^{-1}a^2 \tag{2-23}$$

and finds that the length function w.r.t. the generators x, y is different to that w.r.t. a, b.

This observation is crucial for any attempt to identify algebraic expressions with geometric objects. The same will hold in QFT when identifying operator products.

2.4 Clifford algebras by factorization

Clifford algebras can be approached in a basis free manner which for obvious reasons avoids the problems discussed in the previous section. While generators can be used very conveniently in actual calculations, the strength of the basis free method is to achieve general statements about the structure of Clifford algebras.

Following the procedure which led to the Graßmann algebra, we can introduce an ideal $I_{\mathcal{C}\ell}$ and factor out the Clifford algebra from the tensor algebra $T(V)$. This ideal has to introduce the quadratic form and reads

$$\mathcal{I}_{\mathcal{C}\ell} = \{a \otimes (x \otimes y + y \otimes x) \otimes b - 2g(x,y) a \otimes b | a, b \in T(V), \quad x, y \in V\} \tag{2-24}$$

where $g(x,y)$ is the basis free symmetric polar bilinear form corresponding to Q. Inspection of the elements in this ideal shows that they are not homogeneous and identify elements of different rank. This ideal is not $\mathbb{Z}_n$-graded. Since even- and oddness is preserved by the ideal, it remains to be $\mathbb{Z}_2$-graded.

We arrive at the Clifford algebra via the following factorization

$$\mathcal{C}\ell(V,Q) = \frac{T(V)}{\mathcal{I}_{\mathcal{C}\ell}}\,. \tag{2-25}$$

Following Chevalley [31] (see "The construction and study of certain important algebras") one is able to show that Clifford algebras are universal, which allows to speak about *the* Clifford algebra (up to isomorphy). Existence is also proved in this approach.

The most important and structural interesting observation may be however the identification of $\mathcal{C}\ell$ as a functor. We call a space *reflexive* if its dual has a set of generators of the same cardinality. All finite dimensional spaces are reflexive in this sense. Infinite dimensional spaces are usually not, but if generators are used, we want to have an isomorphism between generators for the spaces V and V^*. Let $\mathcal{H}$ be a reflexive quadratic space, i.e. a pair of a linear space V and a symmetric quadratic form Q. We find that $\mathcal{C}\ell$ is an injective functor from the category (see Chapter 4) of quadratic spaces **Quad** into the category of associative unital algebras **Alg**.

$$\textbf{Quad} \xrightarrow{\quad \mathcal{C}\ell \quad} \textbf{Alg} \tag{2-26}$$

In the same manner we could have introduced a Graßmann functor $\bigwedge$. Functorial investigations would lead us also to the cohomology of these algebras. In fact, we will need the functorial approach later to define the concept of a co-algebra, co-products etc. by a simple duality argument.

2.5 Clifford algebras by deformation – Quantum Clifford algebras

The previous section is to some extend unsatisfactory since it does not allow to *compute* in a plain way. Even the generator and relation method suffers from computational difficulties. It is quite not easy, to Clifford multiply e.g. two extensors u, v. As an example we compute

$$\begin{aligned} e_1 \circ (e_2 \wedge e_3) &= \frac{1}{2} e_1 \circ (e_2 \circ e_3 - e_3 \circ e_2) \\ &= \frac{1}{6}(e_1 \circ e_2 \circ e_3 + e_2 \circ e_3 \circ e_1 + e_3 \circ e_1 \circ e_2 \\ &\qquad - e_1 \circ e_2 \circ e_3 - e_2 \circ e_3 \circ e_1 - e_3 \circ e_1 \circ e_2) \\ &\qquad + \frac{4}{6} g_{12} e_3 - \frac{2}{6} g_{13} e_2 - \frac{4}{6} g_{13} e_2 + \frac{2}{6} g_{12} e_3 \\ &= e_1 \wedge e_2 \wedge e_3 + g_{12} e_3 - g_{13} e_2 \end{aligned} \tag{2-27}$$

which is cumbersome due to the fact that we have to recast exterior products into Clifford products where we can use the (commutator) relations. Finally one has to transform back at the end into the wedge basis of reduced words of the Graßmann basis. Furthermore, that factor 2 occurring in the (anti)commutation relations prevents an application of this mechanism to rings of characteristic 2. Claude Chevalley developed a method which is applicable to this case and which provides an efficient method to evaluate the Clifford product in a Graßmann basis [31]. An emphatic article of Oziewicz [100] generalized Chevalley's method from quadratic forms to bilinear forms. This will be a key point in later applications to QFT.

Chevalley's observation was that it is possible to implement the Clifford algebra as an endomorphism algebra of the Graßmann algebra

$$\mathcal{C}\ell \subset \mathrm{End} \bigwedge V. \tag{2-28}$$

This inclusion is strict. To be able to define an endomorphism on $\bigwedge V$, we have to introduce a dual basis and a dual Graßmann algebra $\bigvee V$. Let $\varepsilon^i(e_j) = \delta^i_j$, where δ^i_j is the Kronecker symbol, and let $\{\mathrm{Id}; \varepsilon^i; \varepsilon^i \vee \varepsilon^j (i < j); \ldots\}$ be a Graßmann co-basis w.r.t. the vee-product. An endomorphism on $\bigwedge V$ can be written as

$$\begin{aligned} R &: \bigwedge V \to \bigwedge V \\ R &= \sum_{IK} R^I{}_K e_I \otimes \varepsilon^K \end{aligned} \tag{2-29}$$

where I, K are multi-indices of ordered basis words (basis monomials).

2.5.1 The Clifford map

Let B be a scalar product $B : V \times V \to \Bbbk$. B is at the same time a map $B : V \to V^*$. The action of the co-vectors ε^i on vectors e_j does form a pairing $< . \mid . >: V^* \times V \to \Bbbk$.

Definiton 2.3 (contraction). *Using the pairing $< . \mid . >_B$, where the scalar product B is used to mediate the adjoint map, a* left (right) contraction $\rfloor$ *($\llcorner$) is defined as*

$$\begin{aligned} &< \varepsilon^i \mid e_j >_B = < \mathrm{Id} \mid B(\varepsilon^i) \rfloor_\delta e_j > = < \mathrm{Id}, e_i \underset{B}{\rfloor} e_j > \\ &< \varepsilon^i \mid e_j >_B = < \varepsilon^i \llcorner_\delta B^{-1}(e_j) \mid_\delta \mathrm{Id} > = < \varepsilon^i \llcorner_{B^{-1}} \varepsilon^j, \mathrm{Id} > \end{aligned} \tag{2-30}$$

Definiton 2.4 (Clifford map). *A* Clifford map $\gamma_x : \bigwedge V \rightarrow \bigwedge V$ *is an endomorphism parameterized by a 1-vector* $x \in V$ *of the following form*

$$\gamma_x = x \underset{B}{\rfloor} + x\wedge \tag{2-31}$$

obeying the following calculational rules ($x, y \in V$*,* $u, v, w \in \bigwedge V$*):*

$$\begin{aligned}
&i) && x \underset{B}{\rfloor} y = B(x,y) \\
&ii) && x \underset{B}{\rfloor}(u \wedge v) = (x \underset{B}{\rfloor} u) \wedge v + \hat{u} \wedge (x \underset{B}{\rfloor} v) \\
&iii) && (u \wedge v) \underset{B}{\rfloor} w = u \underset{B}{\rfloor}(v \underset{B}{\rfloor} w)
\end{aligned} \tag{2-32}$$

where $\hat{}$ *is the main involution* $\hat{} : V \rightarrow -V$*, extended to* $\bigwedge V$*, also called grade involution. One obtains* $\hat{u} = (-1)^{\mathit{length}(u)} u$*.*

We decompose $B = g + F$ into a symmetric part $g^T = g$ and an antisymmetric part $F^T = -F$. The Clifford maps $\{\gamma_{e_i}\}$ of the generators $\{e_i\}$ of V generate the Clifford algebra $\mathcal{C}\ell(V,B)$. Let Id be the identity morphism, we find in a basis free notation

$$\gamma_x\gamma_y + \gamma_y\gamma_x = 2\, g(x,y)\mathrm{Id}\,. \tag{2-33}$$

It is remarkable, that in the anticommutation relation only the symmetric part of B occurs. However, the anticommutators are altered

$$\gamma_x\gamma_y - \gamma_y\gamma_x = 2\, x \wedge y + 2\, F(x,y)\mathrm{Id}\,. \tag{2-34}$$

This shows that the Z_n-grading depends directly on the presence of the antisymmetric part. If we compute a Clifford basis with or without an antisymmetric part F we get ($\gamma^g \in \mathcal{C}\ell(V,g)$, $\gamma^B \in \mathcal{C}\ell(V,B)$)

$$\begin{array}{ccc}
\mathrm{Id} & \qquad & \mathrm{Id} \\
\gamma^g_{e_i}\mathrm{Id} = e_i & & \gamma^B_{e_i}\mathrm{Id} = e_i \\
\gamma^g_{e_i}\gamma^g_{e_j}\mathrm{Id} = e_i \wedge e_j + g_{i,j} & & \gamma^B_{e_i}\gamma^B_{e_j}\mathrm{Id} = e_i \wedge e_j + B_{ij} \\
\text{etc.} & &
\end{array} \tag{2-35}$$

If g is identical zero $g \equiv 0$ we find two different *Graßmann algebras*! One is Z_n-graded w.r.t. the exterior wedge products $\wedge$ while the other is not! It is however possible to introduce a second *dotted wedge* $\dot{\wedge}$, also an exterior product, which is the Z_n-graded product under the presence of the antisymmetric part F.

$$\begin{aligned}
x \,\dot{\wedge}\, y &= x \wedge y + F(x,y)\mathrm{Id} \\
x \,\dot{\wedge}\, y \,\dot{\wedge}\, z &= x \wedge y \wedge z + F(x,y)z + F(y,z)x + F(z,x)y \\
&\text{etc.}
\end{aligned} \tag{2-36}$$

This structure was employed to obtain Hecke algebra representations [51, 5] and is crucial to the compact formulation of Wick's theorem in QFT [47, 50, 56].

2.5.2 Relation of $\mathcal{C}\ell(V,g)$ and $\mathcal{C}\ell(V,B)$

Theorem 2.5 (Wick theorem). *The Clifford algebras $\mathcal{C}\ell(V,g)$ and $\mathcal{C}\ell(V,B)$ are isomorphic as Clifford algebras. The isomorphisms in Z_2-graded.*

Proof: see [47, 50, 57, 56].

Theorem 2.6 (Chevalley [31]). *The opposite Clifford algebra $\mathcal{C}\ell^{op}(V,g)$ of $\mathcal{C}\ell(V,g)$ is isomorphic to $\mathcal{C}\ell(V,-Q)$.*

This can be generalized to

Theorem 2.7. *The opposite Clifford algebra $\mathcal{C}\ell^{op}(V,B)$ of $\mathcal{C}\ell(V,B)$ is isomorphic to $\mathcal{C}\ell(V,-B^T)$.*

Proof: see [60, 50].

One obtains that

$$\mathrm{End} \bigwedge V = \bigwedge V \otimes \bigvee V^* \quad = \mathcal{C}\ell(V,B) \,\hat{\otimes}\, \mathcal{C}\ell(V,-B^T) = \mathcal{C}\ell(V \oplus V, B \oplus -B^T) \tag{2-37}$$

where $\hat{\otimes}$ is a $\mathbb{Z}_2$-graded tensor product. In terms of commutation relations this reads

$$\begin{aligned} \gamma_x\gamma_y + \gamma_y\gamma_x &= 2\,g(x,y) \\ \gamma_x\gamma_y^{op} + \gamma_y^{op}\gamma_x &= 0 \\ \gamma_x^{op}\gamma_y^{op} + \gamma_y^{op}\gamma_x^{op} &= -2\,g(x,y). \end{aligned} \tag{2-38}$$

2.6 Clifford algebras of multivectors

An intriguing approach to Clifford algebras was developed by Oziewicz and will be called *Clifford algebra of multivectors*. This method originated out of a discussion of QF theoretic composite particle calculations [60] which was elevated in [101] to a mathematical setting. We recall this approach here for completeness and because of its extraordinary character and generality.

It was Woronowicz [136, 137] who studied systematically the theory of deformed Graßmann algebras. As we discussed above, Graßmann algebras are obtained by factorization w.r.t. an antisymmetrizer, which projects out all symmetric tensors from tensor algebra. The canonical projection $\pi_\wedge$ maps the tensor product $\otimes$ onto the exterior wedge product $\pi_\wedge(\otimes) \to \wedge$. If one proceeds to deformed symmetries, e.g. Hecke algebras, one obtains deformed Graßmann algebras $\bigwedge_q V$. The presentation of the symmetric algebra reads

$$\begin{aligned} &S_n = < X, \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}\} > \\ &\mathbf{R_1} : s_1^2 = 1 \\ &\mathbf{R_2} : s_i s_j s_i = s_j s_i s_j \\ &\mathbf{R_3} : s_i s_j = s_j s_i \qquad \text{if } |s_i - s_j| \geq 2. \end{aligned} \tag{2-39}$$

X contains $n-1$ generators, s_i. This is a restriction of the Artin braid group, resp. its group algebra, by asserting additionally the relation $\mathbf{R_1}$. The projection operator onto the alternating part reads

$$\pi_\wedge = \frac{1}{n!} \sum_{\text{red. words}} (-1)^{\text{length}(w)} w \tag{2-40}$$

where w runs in the set of $n!$ reduced words. For S_3 we find

$$\pi_\wedge = \frac{1}{3!}(1 - s_1 - s_2 + s_1 s_2 + s_2 s_1 - s_1 s_2 s_1). \tag{2-41}$$

A slight generalization of this setting is to allow a quadratic relation for the transposition which leads to the Hecke algebra

$$\begin{aligned} &H_n = < X, \{\mathbf{R_1}, \mathbf{R_2}, \mathbf{R_3}\} > \\ &\mathbf{R_1} : \tau_1^2 = a\,\tau + b \end{aligned} \tag{2-42}$$

where $\mathbf{R_2}$ and $\mathbf{R_3}$ are still the braid relations. Since the cardinality of the set $Y = \{\text{red. words}\}$ which is generated from the generators $\tau_i \in X$ does not change, one proceeds as above, but has to take care of the additional parameters. Let $a = (1-q)$ and $b = q$, one ends up with a projection operator [5]

$$\pi_{\wedge_q} = \frac{1 - \tau_1 - \tau_2 + \tau_1\tau_2 + \tau_2\tau_1 - \tau_1\tau_2\tau_2}{(1+q+q^2)(1+q)}. \tag{2-43}$$

It is a remarkable fact, that these generators can be found also in an undeformed Clifford algebra if it has a carefully chosen non-symmetric bilinear form [52, 51, 53, 5].

Woronowicz showed that factoring the tensor algebra by such deformed switch generators yields in a functorial way a q-deformed exterior algebra

$$\bigwedge\nolimits_q V = \frac{T(V)}{\mathcal{I}_{\pi_q}} = T(V) \mod \pi_q. \tag{2-44}$$

It should be noted that the relations for such algebras look quite different, involving qs. Moreover, the parameter q has to be treated as a formal variable and deformed exterior algebras have to be built over $\Bbbk[[q]]$.

Oziewicz‘s idea was to study non-grade preserving isomorphisms j of $T(V)$ and their projection under an *ungraded* switch onto exterior algebra. This can be displayed by the following diagram

$$\begin{tikzcd} T(V) \arrow[r, "j"] \arrow[d, "\pi_\wedge"] & T_j(V) \arrow[d, "\pi_\wedge"] \\ \bigwedge V \arrow[r, "\gamma"] & \bigwedge_j V \end{tikzcd} \tag{2-45}$$

The aim is to define the map γ by this diagram and to study the properties of the algebra $\bigwedge_j V$. The main and astonishing outcome is, that if j^2 is a $\mathbb{Z}_2$-graded mapping which respects a filtration

$$\begin{aligned} &j^2 : T(V) \rightarrow T_j(V) \\ &j^2 : T^k(V) \rightarrow T^k_j(V) \oplus T^{k-2}_J(V) \oplus \ldots \oplus \begin{cases} T^1_j(V) & \text{if } k \text{ is odd} \\ T^0_j(V) & \text{if } k \text{ is even} \end{cases} \end{aligned} \tag{2-46}$$

Oziewicz proved that $\bigwedge_j V$ is a Clifford algebra w.r.t. an arbitrary bilinear form induced by j^2. Since we have no occasion to follow this interesting path, the reader is invited to consult the original work [101]. We will deliver an example which provides some evidence that the above described mechanism works.

Example: Let $a, b \in T^1(V)$ and $j^2 : V \otimes V \rightarrow T(V)$ be defined as $j^2(a \otimes b) = a \otimes b + B_{ab}$ where B is an arbitrary bilinear form. We compute the above given commutative diagram on these elements

$$\begin{tikzcd} (a \otimes b) \arrow[r, "j^2"] \arrow[d, "\pi_\wedge"] & a \otimes b + B_{ab} \arrow[d, "\pi_\wedge"] \\ a \wedge b = \frac{1}{2}(a \otimes b - b \otimes a) \arrow[r, "\gamma^B"] & a \wedge b + B_{ab} = \frac{1}{2}(a \otimes b - b \otimes a) + B_{ab} \end{tikzcd} \tag{2-47}$$

If γ is interpreted as the action of a on b it constitutes a Clifford map $\gamma_a b = a \wedge b + B_{ab}$. The general case is given in Oziewicz [101].

Relevant to our consideration is that this construction can be interpreted as a product mutation or the other way around a homomorphism of algebras. Let Γ be the map γ extended to $\bigwedge V$, we find

$$\begin{aligned} \Gamma : \bigwedge V \rightarrow \bigwedge_j V \cong \mathcal{C}\ell(V, B) \\ \Gamma(a \wedge b) = \Gamma(a) \circ \Gamma(b) \end{aligned} \tag{2-48}$$

where $\circ$ is the product of the new algebra, in our case a Clifford product. An analogous mechanism was used by Brouder to introduce renormalized time-ordered products in QFT.

A further remarkable fact is that one can discuss deformation *versus* quantization. It might be even surprising that a Clifford algebra can be considered as an exterior algebra w.r.t. a different $\mathbb{Z}$-grading. This is obtained from the identification $\bigwedge_j V \cong \mathcal{C}\ell(V, B)$. Such an outcome depends strongly on the properties of j^2. Oziewicz's method is much more general and various algebras may be generated along this lines. It is obvious that such a construction holds for the symmetric algebras and Weyl algebras also.

2.7 Clifford algebras by cliffordization

Studying cliffordization is a major aspect of this treatise. We postpone its precise elaboration to later chapters. In this section we discuss cliffordization in a non-technical way and try to highlight the advantages of cliffordization and to make contact to some notions from the group theory. This will help to recognize the fundamental nature of cliffordization not only in our case.

The Clifford map γ_x introduced by Chevalley is a mapping

$$\gamma_x : V \times \bigwedge V \to \bigwedge V \tag{2-49}$$

and thus quite asymmetric in the structure of its factors. Stressing an analogy, we will call the process induced by the Clifford map as *Pieri formula* of Clifford algebra. In the theory of the symmetric group (alternating groups included) a Pieri formula allows to add a single box to a standard Young tableaux and gives the result expanded into such standard tableaux, see e.g. [61]. Denote a partition of the natural number n into k parts as $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$, with $\sum \lambda_i = n$. Young operators can be constructed which are projection operators allowing a decomposition of the representation space. The formulas which allow to add one box (possibly in each row) to a Young tableau is a Pieri formula

$$Y^1 \circ Y^{(\lambda_1 \ldots \lambda_k)} = \sum_K a_K Y^{\lambda_K} \tag{2-50}$$

where K runs over all partitions of the standard Young tableaux obtained by adding the box. The crucial point is to have an explicite rule to calculate the coefficients a_K in this expansion, the *branching rule* and *branching coefficients*. In the case of a Clifford map these coefficients emerge from the contractions and the involved bilinear form.

Recursive application of the Pieri formula allows to calculate products of arbitrary Young tableaux. A *closed* formula for such a product is called a *Littlewood-Richardson rule*. The question is, if such a formula can be given for Clifford algebras too. The affirmative answer was given by Rota and Stein [119, 118].

Using the co-product which we introduced by employing shuffles of $(r, s-r)$-splits one can give the following formula for a Clifford product of two reduced words

$$u \circ v = B^{\wedge}(u_{(2)}, v_{(1)})\, u_{(1)} \wedge v_{(2)} \tag{2-51}$$

where $B^{\wedge} : \bigwedge V \times \bigwedge V \to \Bbbk$ is the extension of $B : V \times V \to \Bbbk$ by exponentiation. The product is extended to $\bigwedge V$ by bilinearity. Hence we identify $B^{\wedge}(u_{(2)}, v_{(1)})$ with the branching coefficients.

It would be misleading to recognize 'cliffordization' as closely tied to 'Clifford' algebras. Rota and Stein showed in Refs. [119, 118] that this is in fact a general mechanism and that e.g. the Littlewood-Richardson rule emerges as a special case. Cliffordization provides a direct and computational very efficient approach to various product formulas of deformed structures. The language of cliffordization is that of Hopf gebras.

The above example using Young tableaux is not far away from our topic. Representation theory of $gl_n(\mathbb{C})$ is closely related to this topic. Therewith related irreducible representations of the symmetric group are called Specht modules or Schur modules if finite representations over $\mathbb{C}$ are considered. If one fills Young diagrams not by numbers but by vectors, the resulting spaces are the Schubert varieties, which are extremely useful in algebraic geometry. Graßmannians, flag manifolds and cohomological aspects can be treated along this route.

Given the variety of approaches to 'Quantum Clifford Algebras' it is clear, that we have to study cliffordization as the most general and promising tool for a great bunch of interesting mathematical and physical problems. Especially quantum field theory will benefit extraordinarily from cliffordization.

2.8 Dotted and un-dotted bases

It is a triviality that one can choose various bases to span the linear space underlying an algebra. In our case, it is convenient to use reduced words w.r.t. the wedge product $\wedge$, the Clifford product $\circ$ or the dotted wedge product $\dot\wedge$ which leads to bases of the following type

$$
\begin{array}{rlll}
i) & GB = \{\mathrm{Id}; e_i; e_i \wedge e_{j_{i<j}}; \ldots\} & \text{Graßmann basis} & \\
ii) & CB = \{\mathrm{Id}; e_i; e_i \circ e_{j_{i<j}}; \ldots\} & \text{Clifford basis} & \\
iii) & dGB = \{\mathrm{Id}; e_i; e_i \,\dot\wedge\, e_{j_{i<j}}; \ldots\} & \text{dotted Graßmann basis.} & \text{(2-52)}
\end{array}
$$

We will investigate a few cases where a choice of the basis leads to a different outcome.

2.8.1 Linear forms

A Clifford algebra comes with a unique linear form. We have identified the scalars by the unit map $\eta : \Bbbk \to \mathrm{Id} = V^0$. It is now convenient to introduce the inverse mapping ϵ such that $\epsilon \circ \eta = \mathrm{Id}_{\Bbbk}$

$$\Bbbk \xrightarrow{\quad\eta\quad} \mathcal{C}\ell(V,B) \xrightarrow{\quad\epsilon\quad} \Bbbk \tag{2-53}$$

Technically speaking, the linear form ϵ comes up with the coefficient of Id in the expansion of an element u in terms of the basis

$$
\begin{aligned}
u &= u^0\mathrm{Id} + u^i e_i + u^{ij} e_i \wedge e_j + \ldots \\
\epsilon(u) &= u^0
\end{aligned} \tag{2-54}
$$

But, this outcome depends strongly on the basis chosen to expand an element. Converting an expansion from one product to another will change the value of the linear form. Let us take

$u = u^0\mathrm{Id} + u^i e_i + u^{ij} e_i \wedge e_j$ and compute

$$
\begin{aligned}
&GB & u &= u^0\mathrm{Id} + u^i e_i + u^{ij} e_i \wedge e_j \\
& & \epsilon(u) &= u^0 \\
&CB & u &= u^0\mathrm{Id} + u^i e_i + u^{ij} e_i \circ e_j \\
& & &= u^0\mathrm{Id} + u^i e_i + u^{ij}(e_i \wedge e_j + B_{ij}) \\
& & \epsilon(u) &= u^0 + u^{ij} B_{ij} \\
&dGB & u &= u^0\mathrm{Id} + u^i e_i + u^{ij} e_i \dot{\wedge} e_j \\
& & &= u^0\mathrm{Id} + u^i e_i + u^{ij}(e_i \wedge e_j + F_{ij}) \\
& & \epsilon(u) &= u^0 + u^{ij} F_{ij}.
\end{aligned}
\tag{2-55}
$$

It is thus convenient to introduce a unique linear form $\epsilon^{\wedge}$, ϵ°, and $\epsilon^{\dot{\wedge}}$ for every basis. These are different linear forms and their appearance is important in quantum physics. Moreover, we saw that $\mathcal{C}\ell(V,g)$ and $\mathcal{C}\ell(V,B)$ are isomorphic as $\mathbb{Z}_2$-graded algebras, but they possess different canonical linear structures. The isomorphy, mediated by the Wick theorem, is nothing but a change in the product from the wedge to the dotted wedge and vice versa. Of course, one could introduce a new unit map $\eta^{\dot{\wedge}}$ or η° to avoid these problems, however, we will see that there are other obstructions which prevent this.

2.8.2 Conjugation

The main involution of a Graßmann or Clifford algebra was given as the the map $\hat{}\,: V \to -V$ extended to $\bigwedge V$. In terms of reduced words, this reads

$$
\hat{u} = (-1)^{\text{length}(u)}\, u. \tag{2-56}
$$

It is now obvious, that the length function depends also on the chosen expansion to a basis, if the basis is not of the same $\mathbb{Z}_2$-grade (parity). Since we do not investigate such supersymmetric transformations here, all bases behave similar under this involution.

2.8.3 Reversion

This changes if we come to the reversion $\tilde{}\,: \bigwedge V \to \bigwedge V$. Reversion is an anti algebra homomorphisms

$$
\begin{aligned}
(u \wedge v)\tilde{} &= \tilde{v} \wedge \tilde{u} \\
\tilde{}\,|_{\mathrm{Id}\oplus V^1} &= \mathrm{Id}_{\mathrm{Id}\oplus V^1}.
\end{aligned}
\tag{2-57}
$$

This operation is quite sensible to the chosen basis, as we will exemplify once more on the element $u = u^0\mathrm{Id} + u^ie_i + u^{ij}e_i \wedge e_j$.

$$
\begin{aligned}
GB \qquad\qquad & u = u^0\mathrm{Id} + u^ie_i + u^{ij}e_i \wedge e_j \\
& \tilde{u} = u^0\mathrm{Id} + u^ie_i - u^{ij}e_i \wedge e_j \\
CB \qquad\qquad & u = u^0\mathrm{Id} + u^ie_i + u^{ij}e_i \circ e_j \\
& \quad = u^0\mathrm{Id} + u^ie_i + u^{ij}(e_i \wedge e_j + B_{ij}) \\
& \tilde{u} = u^0\mathrm{Id} + u^ie_i - u^{ij}(e_i \wedge e_j - B_{ij}) \\
& \quad = (u^0 - 2u^{ij}B_{ij})\mathrm{Id} + u^ie_i - u^{ij}e_i \circ e_j \\
dGB \qquad\qquad & u = u^0\mathrm{Id} + u^ie_i + u^{ij}e_i \dot{\wedge} e_j \\
& \tilde{u} = u^0\mathrm{Id} + u^ie_i - u^{ij}(e_i \wedge e_j - F_{ij}) \\
& \quad = (u^0 + 2u^{ij}F_{ij})\mathrm{Id} + u^ie_i - u^{ij}e_i \dot{\wedge} e_j.
\end{aligned} \tag{2-58}
$$

Since the reversion is needed to form spinor inner products, this is an outcome of major importance. Also Clifford-Lipschitz, **pin** and **spin** groups will be altered by this mechanism. Indeed we have been able to employ this type of transformation to study q-spin groups, and Hecke algebras [52, 51, 5].

Note, that one is once more able to define a reversion w.r.t. any product which was chosen to build the reduced words or even a different one. In the above calculation we used reversion w.r.t. the wedge product. Regarding the group structures coming with Clifford algebras, it might be convenient to use the reversion w.r.t. the Clifford product. Reversion may be called *Graßmann reversion*, *dotted Graßmann reversion* or *Clifford reversion* to indicate w.r.t. which product it acts as antihomomorphism.

Given a reversion, say w.r.t. the Clifford product, one finds an exterior product which is stable under reversion in that sense that is does not pick up additional terms of a different grade. This is the dotted wedge product. This will justify the identification of dotted wedge products with normal-ordered operator products while the undotted wedge will be related to time-ordered operator product. This relations was established in [47, 60, 49, 50, 56]. Clifford reversion acts as:

$$
\begin{aligned}
(e_i \wedge e_j)\tilde{} &= -e_i \wedge e_j + 2F_{ij} \\
(e_i \dot{\wedge} e_j)\tilde{} &= -e_i \dot{\wedge} e_j.
\end{aligned} \tag{2-59}
$$

Chapter 3

Graphical calculi

3.1 The Kuperberg graphical method

3.1.1 Origin of the method

In 1991, Kuperberg introduced a graphical method to visualize tensorial equations [84]. His method received some recognition, e.g. [76, 77], since he derived a valuable set of lemmas and theorems in the course of calculating an invariant for 3-manifolds. While the first paper mentioned above deals with involutory Hopf algebras, the second paper [85] generalized the method to the non-involutory case.

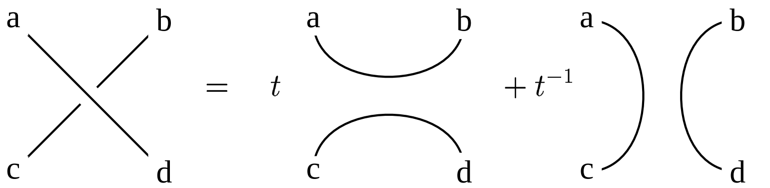
Tensors appear quite naturally at nearly any place in physics. Kuperberg's starting point is the theory of *state models* M. Such models consist of a commutative ring **R** (usually the field $\mathbb{C}$) and a bi-partite graph G, the *connectivity graph,* whose vertices are labelled as *atoms* and *interactions*; a set S_A for each atom A, called the *state set* of A; and a function $\omega_I : A_1 \times A_2 \times \ldots \times A_n \rightarrow \mathbf{R}$ for each interaction I (where $A_1, \ldots, A_n$ are *neighbours* of I), called *weight function* or the *Boltzmann weights* of I. A *state* of M is a function s on the atoms of M such that $s(A) \in S_A$. The *weight* $\omega(s)$ of a state s is defined as the product of the ω_I's evaluated at the state s when this product converges, and in particular when G is finite. Finally, the *partition function* $Z(M)$ is defined to be the sum of $\omega(s)$ over all states s when this sum converges, and in particular when all state sets are finite.

There are more types of models like face type models or ice type models, which however can be handled along the same lines [12].

In topology, a state model is connected to *knots* and *links*, if a projection P of a link is given, one declares the arcs between *crossings* to be atoms and the crossing (which contains the information which arc crosses over and which one under the other one) themselves to be interactions. Crossings may also be called *scatterings* which is derived from a particle interpretation of these models. A *weight function* on states a, b, c, d may be defined as

$$\omega(a, b, c, d) = t\delta(a, b)\delta(c, d) + t^{-1}\delta(a, c)\delta(b, d) \tag{3-1}$$

where t is chosen so that $n = -(t^2 + t^{-2})$, and $\delta(a, b) = 1$ when $a = b$ and 0 otherwise. This state model is a 'link covariant' called the Kauffman bracket [75], which is essentially the Jones polynomial up to normalization. Pictographically the crossing is written as



(3-2)

This diagram is also an instance of a *skein relation* which allows to cut knots and links into smaller and more elementary objects. It can easily be checked, that if $\dim V = n$ one finds the inverse scattering from the substitution $t \to t^{-1}$. Moreover this scattering is a braid, i.e. it satisfies the braid relations of the Artin braid group.

For our purpose, we review Kuperberg's graphical method, which is then afterwards compared with the method using tangles [139, 88, 89]. Moreover, we are interested in some basic results derived by Kuperberg, e.g. Lemma 3.1. [84]. A further interesting result, which will be discussed in a subsequent chapter, is the fact that quantum Clifford Hopf gebras and quantum Graßmann Hopf gebras provide counterexamples to Kuperberg's Lemma 3.2., which has to be reformulated to hold on *non-interacting* Hopf gebras. This analysis will allow us to describe a distinction of *interacting* and *non-interacting* products, co-products and Hopf gebras.

3.1.2 Tensor algebra

Tensorial equations use an index notation which is common in physics and is mainly used in hydrodynamics, electrodynamics, special and general relativity. The invariant objects like *vectors* and *tensors* are displayed via their components w.r.t. a (commonly not written down) basis,

$$T = T^{kl}{}_{ij} e_k \otimes e_l \otimes \epsilon^i \otimes \epsilon^j \quad \mapsto \quad T^{kl}{}_{ij}. \tag{3-3}$$

A basis is assumed, but it needs not even to be a holonomic basis on a manifold. A thoughtful introduction to abstract vectors and the usage of indices can be found in Penrose Rindler [106, 107].

Let now V be a module (vector space), elements of V are written as v^a. The dual space V^* of linear forms gives rise to elements ω_b acting on the v^a. In abstract index notation, a and b are 'placeholders of names' of a vector and a linear form (co-vector) respectively, we will also write the index $a, b, c, \ldots$ directly to denote a vector or co-vector there using Greek letters. On the other hand using letters as indices of kernel symbols, e.g. v^a, one assumes a vector to be a n-tuple of $\Bbbk$-numbers, then the v^a are simply the components of the vector. With Kuperberg we will from now on assume a canonical basis in V and V^* denoted by $\{e_i\}$ and $\{\epsilon^j\}$. All spaces are assumed to have finite dimension, i.e. the index set $\mathcal{I}$ is of finite order $\#\mathcal{I} < \infty$.

Vectors are then described as tuples of coefficients, indexed by the 'names' of the basis vectors:

$$v \cong (v^1, \ldots, v^n). \tag{3-4}$$

The components of v can be obtained by applying the canonical co-vectors ϵ^i on v, assuming the relation

$$\epsilon^i(e_j) = \delta^i_j, \tag{3-5}$$

which fixes $\#\mathcal{I}$ linear forms $\epsilon^i \in V$. Obviously we have

$$\epsilon^i(v) = \epsilon^i(v^j e_j) = v^j \delta^i_j = v^i, \tag{3-6}$$

where the Einstein summation convention is in force between upper and lower indices.

An endomorphism $S : V \to V$ is an element of lin-Hom $(V, V) \cong V \otimes V^*$ and it has therefore the index structure

$$S^a{}_b \cong S = S^a{}_b e_a \otimes \epsilon^b. \tag{3-7}$$

The action of an endomorphism is translated (via summation convention) in this method into *matrix multiplication* $\bullet$ of the coefficients.

$$\begin{aligned} S \bullet v &= (S^a{}_b e_a \otimes \epsilon^b)(v^c e_c) \\ &= S^a{}_b v^c e_a \epsilon^b(e_c) \\ &= S^a{}_b v^c e_a \delta^b_c \\ &= S^a{}_b v^b e_a, \end{aligned} \tag{3-8}$$

which reads after dropping the basis vectors as usual

$$v'^a = S^a{}_b v^b. \tag{3-9}$$

We have to distinguish four type of maps, which are different in their index structure:

$$\begin{array}{lcl} S^a{}_b & V \xrightarrow{S} V & \text{endomorphism on } V \\ T_a{}^b & V^* \xrightarrow{T} V^* & \text{endomorphism on } V^* \\ B_{ab} & V \xrightarrow{B} V^* & \text{scalar product} \\ C^{ab} & V^* \xrightarrow{C} V & \text{co-scalar product.} \end{array} \tag{3-10}$$

Note that the usual symmetry types can be established on type-changing operations having two indices of the same type, i.e. B or C:

$$\begin{aligned} B_{(ab)} &= g_{ab} = \frac{1}{2}(B_{ab} + B^T_{ab}), \quad B^T_{ab} = B_{ba} \\ B_{[ab]} &= A_{ab} = \frac{1}{2}(B_{ab} - B^T_{ab}), \end{aligned} \tag{3-11}$$

where we have introduced the Bach-brackets common in tensor calculus of general relativity. Vector (component) indices are called *contra-variant* and co-vector (component) indices are called *co-variant*. This notion reflects the fact that under a (linear) change of the basis the vector coefficients transform in an inverse (contra $\cong$ against) way as the basis itself. Co-vector indices transform covariant (co $\cong$ with). Hence this notion resorts to the invariance of the vectors (tensors)

$$\begin{aligned} v = v^a e_a &= (v^a {T^{-1}}_a{}^b)(T_b{}^c e'_c) \\ &= v'^b e'_b. \end{aligned} \tag{3-12}$$

The attentive reader will have noticed that T^{-1} acts from the right, since it has to mimic a map

$$V^* \xrightarrow{\quad T \quad} V^* \tag{3-13}$$

which would have the index structure $T^b{}_a$. This shows directly the pitfall to look at coefficients (and tuples of coefficients) as constituting 'vectors', but see e.g. Hilbert [71]. The v^b are simply elements of the number field (or a commutative ring $\mathbf{R}$) and obey *no* vectorial transformation law at all. Tensor calculus, by omitting the basis, shifts in a peculiar manner the vector character of the object $v = v^a e_a$ to the index (position) of the component.

We introduce some more notations. A tensor is said to have *step* n if it has n indices. The terms *rank*, *degree* or *grade* are sometimes used also. It is said to have *type* (p, q) if it has p contravariant and q covariant indices.

Basic actions with tensors are:

i) Tensors may be added if and only if they have the same index structure (type) including the names of these indices

$$A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} + B^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} = C^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}. \tag{3-14}$$

ii) Tensors may be multiplied. The product tensor picks up all indices in their mutual order and gets a new kernel symbol

$$\begin{aligned} v^a w^b &= U^{ab} \\ A^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} B^{k_1,\ldots,k_n}_{l_1,\ldots,l_n} &= C^{i_1,\ldots,i_r,k_1,\ldots,k_n}_{j_1,\ldots,j_s,l_1,\ldots,l_n}. \end{aligned} \tag{3-15}$$

iii) Factor switching (transposition) is given by the map $a \otimes b \xrightarrow{T} b \otimes a$ of adjacent or non-adjacent indices of the same type. This device allows to speak about symmetry, i.e.

$$g_{(ab)} = g_{ba} = g^T_{ab}, \quad T : V^* \otimes V^* \to V^* \otimes V^* \tag{3-16}$$

is a symmetric tensor. This map will be called *switch* if adjacent indices (elements) are interchanged.

iv) The canonical map from $V^* \otimes V$ into $\Bbbk$ which is called *trace map* or *evaluation map* is denoted by repeated indices. Note that this map implicitly uses the isomorphism

$$V \xrightarrow{\quad\star\quad} V^*, \tag{3-17}$$

which is called Euclidean dual isomorphism by Saller [123]. This map is needed to establish the correspondence

$$e_i \xrightarrow{\quad\star\quad} \epsilon^i \tag{3-18}$$

and is assumed to be bijective allowing to establish $\star^{-1}$

$$\epsilon^i \xrightarrow{\quad\star^{-1}\quad} e_i \tag{3-19}$$

Examples are $\omega_a v^a$ which is the value of ω at the point v, $S^b{}_a v^a$ is the action of the endomorphism S on v, $S^a{}_a$ is the *trace* of S.
The trace or evaluation is commonly called *contraction* in tensor calculus.

Free (open) i.e. uncontracted indices will be called *boundary* indices, while contracted indices are called *inner*.

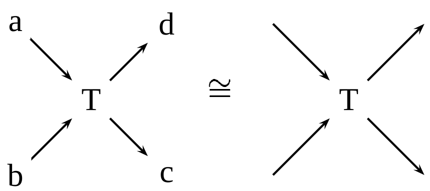
3.1.3 Pictographical notation of tensor algebra

Kuperberg's translation of tensor equations into a graphical language is now as follows:

i) Every tensor is represented by its kernel symbol. Scalars are usually not drawn at all, since they have no connectivity, i.e. no open or boundary indices. In graphical terms this corresponds to in- or out-going arrows.

ii) Every contravariant index is represented as an arrow pointing towards the kernel symbol.

iii) Every covariant index is represented as an arrow pointing away from the kernel symbol.
Example: A tensor of step 4 and type $(2,2)$ i.e. T^{ab}_{cd} is iconographically represented as:



(3-20)

An endomorphism is given as $S^a{}_b$

$$\to S \to . \tag{3-21}$$

A vector v or a co-vector ω appears as source or sink of an arrow

$$v \to, \qquad \to \omega. \tag{3-22}$$

iv) Contraction of tensors translates into connecting diagrams.
Example: $S^a{}_b v^b$ the endomorphic product as

$$v \to S \to . \tag{3-23}$$

The trace $S^a{}_a$ and $\mathrm{Id}^a{}_a = \mathrm{tr}(\mathrm{Id})$ are depicted as

(3-24)

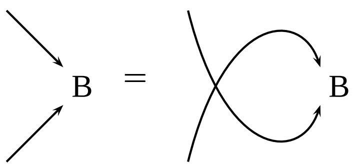
where these diagrams having no in- or outgoing arrows represent scalars, i.e. the trace of S and the trace of Id which is $\dim V$.

v) Arrows are allowed to and will cross.

If boundary arrows are not labelled there is an ambiguity in their re-labelling. However, if we adopt the rule that external lines will be named counter clockwise starting at the top-left arrow, and that arrows of diagrams which will be subjected to an equality have to end at equal places, this ambiguity is removed.
Examples: A linear equation results in (l.h.s. vector equation, r.h.s. equation for a scalar coefficient)

$$v \to S \to = \ v' \to \quad \Rightarrow \quad v \to S \to \epsilon = v' \to \epsilon \tag{3-25}$$

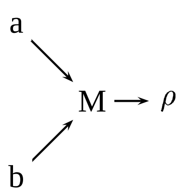
and the symmetry of a bilinear form translates into



$$\Rightarrow \quad B_{ab} = B_{ba}. \tag{3-26}$$

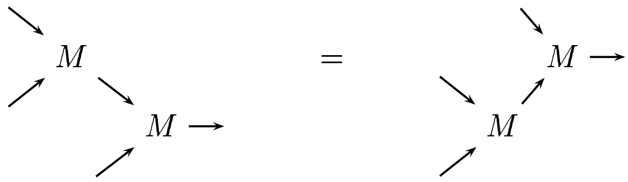
3.1.4 Some particular tensors and tensor equations

The multiplication of an algebra can be described as tensor of rank 3 with valence $1, 2$. The components M^i_{jk} are called multiplication coefficients and $\{M^i_{jk}\}$ is the multiplication table which uniquely defines the product structure of the algebra.



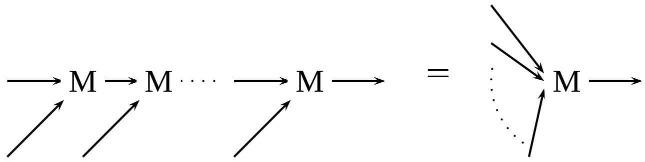
$$\cong \quad \rho_l M^l_{ij} a^i b^j. \tag{3-27}$$

Note that ρ is a co-vector while a, b are vectors and the equation holds between scalars. Associativity is represented by



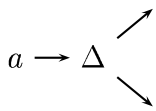
(3-28)

which permits one to drop braces and to condense the diagrams as follows



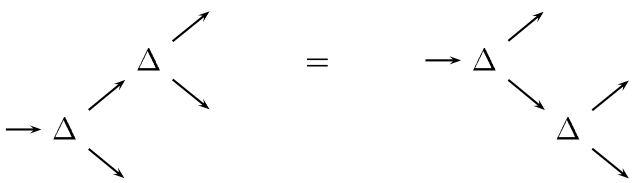
(3-29)

Co-algebras and co-gebras can be defined by a certain categorial duality described in some detail in the next chapter. In the Kuperberg graphical calculus, this results in the reversion of the arrows and obviously in renaming of the structure elements. As an example, we can define a co-product Δ which splits up a single line (module) into a tensor product of two lines (modules)



$$\Leftrightarrow \quad \Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}. \tag{3-30}$$

We have employed the so called Sweedler notation [130] to indicate the elements of the co-product in the first and second tensor slot. As a rule for translation, one might associate the terms $a_{(i)}$ in counter clockwise order of arrows from left to right in the tensor products. In our case $a_{(1)}$ would be the lower outgoing arrow etc. Internal names (indices) have no particular meaning at all. The co-product is also called diagonalization, since some well recognized co-products, e.g. that of groups, simply double the entry element $\Delta(a) = a \otimes a$. The associativity of co-products, i.e. co-associativity, is derived from the associativity if we replace $\rightarrow \Leftrightarrow \leftarrow$ and $M \Leftrightarrow \Delta$ and it reads:



(3-31)

If a product has an unit, this is pictographically represented as

$$\begin{tikzcd}[row sep=small] \eta \arrow[dr] & & & {} \arrow[dr] & & & & \\ & M \arrow[r] & {} = & & M \arrow[r] & {} = & {} \arrow[r] & {} \\ {} \arrow[ur] & & & \eta \arrow[ur] & & & & \end{tikzcd} \tag{3-32}$$

and dualizing yields the definition of a counit ϵ

$$\begin{tikzcd}[row sep=small] & & \epsilon & & & {} & & \\ {} \arrow[r] & \Delta \arrow[ur] \arrow[dr] & = & {} \arrow[r] & \Delta \arrow[ur] \arrow[dr] & = & {} \arrow[r] & {} \\ & & {} & & & \epsilon & & \end{tikzcd} \tag{3-33}$$

A further prominent structure element is the antipode S, an endomorphism, which, if it exists, fulfils the following defining relations

$$\begin{tikzcd}[column sep=small] & & S \arrow[dr] & & & & & & & & & & & \\ {} \arrow[r] & \Delta \arrow[ur] \arrow[rr, bend right] & & M \arrow[r] & {} = & {} \arrow[r] & \epsilon \quad \eta \arrow[r] & {} = & {} \arrow[r] & \Delta \arrow[rr, bend left] \arrow[dr] & & M \arrow[r] & {} \\ & & & & & & & & & & S \arrow[ur] & & \end{tikzcd} \tag{3-34}$$

In other words, S is the left and right *convolutive* inverse of the linear identity $\mathrm{Id} \in \mathrm{End}\, V$. Having the Hopf algebra as our goal in mind, it is convenient to introduce a further relation, which is the graphical counterpart of the fact that M is asserted to be a co-algebra morphism and Δ is an algebra morphism. This can be postulated as an axiom, which will result in a constraint to choose M and Δ or can be checked to be true or false for an arbitrary given pair of structure tensors (M, Δ).

$$\begin{tikzcd}[row sep=small] {} \arrow[dr] & & & {} & & & & & \\ & M \arrow[r] & \Delta \arrow[ur] \arrow[dr] & = & {} \arrow[r] & \Delta \arrow[r] \arrow[ddr] & M \arrow[r] & {} \\ {} \arrow[ur] & & & {} & & & & \\ & & & & {} \arrow[r] & \Delta \arrow[r] \arrow[uur] & M \arrow[r] & {} \end{tikzcd} \tag{3-35}$$

Note that a crossing of arrows occurs in the right hand side of this equation. This will be allowed and it is described in more detail below.

The notion of a Hopf algebra is equivalent to the assertion of the relations in Eqns. 3-31 to 3-35. Indeed, some rules to manipulate the Kuperberg graphs have to be given since the mere notion of *calculating* with them means that there are rules to manipulate them.

Hopf algebras have been derived from some properties of groups and group manifolds [72, 94]. For groups one obtains the structure relations

$$
\begin{array}{ccc}
g & & \\
& \searrow & \\
& & M \longrightarrow \\
& \nearrow & \\
h & &
\end{array}
\;=\; gh \longrightarrow
\qquad\qquad
\begin{array}{ccc}
& & \\
& \nearrow & \\
g \longrightarrow \Delta & & \\
& \searrow & \\
& &
\end{array}
\;=\;
\begin{array}{c}
g \longrightarrow \\
\\
\\
\\
g \longrightarrow
\end{array}
\tag{3-36}
$$

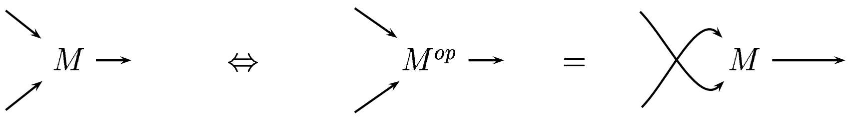
$$
g \longrightarrow S \longrightarrow \;=\; g^{-1} \longrightarrow
$$

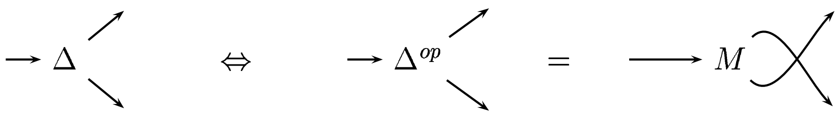
having the nice property that Hopf algebra morphisms induce as restrictions morphisms of the underlying groups.

3.1.5 Duality

There are different ways to introduce new Hopf algebra structure tensors related to the given one. Indeed, we have already used the fact that we can exchange by categorial duality $\Delta \Leftrightarrow M$, $\epsilon \Leftrightarrow \eta$, $\rightarrow \Leftrightarrow \leftarrow$ etc. This Hopf algebra is denoted by H^* and H may be denoted as H_* for topological reasons [72, 94].

A further possibility to introduce new structure elements related with the old ones is to introduce opposite algebras and opposite co-gebras, which are given via


(3-37)


(3-38)

$$
\longrightarrow S \longrightarrow \quad \Leftrightarrow \quad \longrightarrow S^{op} \longrightarrow \tag{3-39}
$$

If the opposite algebra and co-algebra structures have units η^{op} and counits ϵ^{op} depends on the crossing of arrows, but are taken usually to be the same units η and counits ϵ as in the untwisted case, which is an assumption about the crossing.

3.1.6 Kuperberg's Lemma 3.1.

In 1991 [84] Kuperberg introduced so called *ladder* diagrams, according to their graphical representation, which are extremely useful for proving further identities. A further Lemma from this important paper will be considered below.

Lemma 3.1 (Kuperberg). *The tensors of (bi-associative, bi-unital) Hopf objects*

$$\begin{array}{c} \rightarrow M \rightarrow \\ \uparrow \\ \rightarrow \Delta \rightarrow \end{array} \qquad\qquad \begin{array}{c} \rightarrow M \rightarrow \\ \uparrow \\ \leftarrow \Delta \leftarrow \end{array} \tag{3-40}$$

when viewed as vector space endomorphisms of $H \otimes H$ and $H \otimes H^*$, are invertible.

Proof:

$$\begin{array}{ccccc} \rightarrow M \rightarrow & M \rightarrow \\ \uparrow & \uparrow \\ S & \vert \\ \uparrow & \vert \\ \rightarrow \Delta \rightarrow & \Delta \rightarrow \end{array} \quad = \quad \begin{array}{ccc} & & \rightarrow M \rightarrow \\ & & \uparrow \\ \Delta & \rightarrow S \rightarrow & M \\ \uparrow & & \\ \rightarrow \Delta \rightarrow & & \end{array} \quad = \quad \begin{array}{cc} & \rightarrow M \rightarrow \\ & \uparrow \\ \epsilon & \eta \\ \uparrow & \\ \rightarrow \Delta \rightarrow & \end{array} \quad = \quad \begin{array}{c} \longrightarrow \\ \\ \\ \\ \longrightarrow \end{array} \tag{3-41}$$

$$\begin{array}{ccccc} \rightarrow M \rightarrow & M \rightarrow \\ \uparrow & \uparrow \\ S & \vert \\ \uparrow & \vert \\ \leftarrow \Delta \leftarrow & \Delta \leftarrow \end{array} \quad = \quad \begin{array}{ccc} & & \rightarrow M \rightarrow \\ & & \uparrow \\ \Delta & \rightarrow S \rightarrow & M \\ \uparrow & & \\ \leftarrow \Delta \leftarrow & & \end{array} \quad = \quad \begin{array}{cc} & \rightarrow M \rightarrow \\ & \uparrow \\ \epsilon & \eta \\ \uparrow & \\ \leftarrow \Delta \leftarrow & \end{array} \quad = \quad \begin{array}{c} \longrightarrow \\ \\ \\ \\ \longleftarrow \end{array} \tag{3-42}$$

The first equality is due to associativity, the second holds because of the antipode axiom while the third reflects the unit and counit properties.

3.2 Commutative diagrams versus tangles

3.2.1 Definitions

We do not intend to go into details of category theory, hence the interested reader may consult e.g. Mac Lane [90], whom we follow in our presentation. Since some notions of category theory are, however, frequently used in physics, we want to give definitions for the most frequently used terms which are also freely used in this work. Especially the literature defining Hopf algebras is full of *commutative diagrams*, and uses the notion of *categorial duality*, i.e. reversing of arrows in diagrams. At the same time, we introduce *tangles*. Tangles will be seen to be opposite (might be also called dual, but should not be confused with categorial duality) to commutative diagrams in a certain sense.

Categories and functors are most often described by graphical methods. It is therefore appropriate to define a *metagraph* which consists of *objects* $a, b, c, \ldots$ and *arrows* $f, g, h, \ldots$. The arrows depict morphisms of some structure of the objects. Every arrow has a *domain* or *source* and a *codomain* or *target*. We have thus

$$a = \operatorname{dom} f, \qquad b = \operatorname{codom} f. \tag{3-43}$$

A morphism f can be graphically represented in two ways:

$$f\ :\ a \to b \qquad\qquad \text{or} \qquad\qquad a \xrightarrow{\quad f \quad} b. \tag{3-44}$$

A finite graph may be composed from such objects and arrows. A *metacategory* is a metagraph with two additional operations. The *identity* which assigns to each object the morphism $\mathrm{Id}_a\ :\ a \to a$ and the composition, which assigns to each pair of morphisms f, g having $\mathrm{dom}\, f = \mathrm{codom}\, g$ the *composite* morphism $h = g \circ f$. One finds that $\mathrm{dom}\, h = \mathrm{dom}\, f$ and $\mathrm{codom}\, h = \mathrm{codom}\, g$. This operation can be most clearly displayed by a *commutative diagram* which relates the arrows as follows

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h = g \circ f"'] & & c \end{tikzcd} \tag{3-45}$$

The diagrammatic description contains full information about all arrows, their domains and co-domains, and objects involved in the transformations. The diagram is called *commutative* since we end up with the result c if we take either route $g \circ f$ or h. The composition of morphisms, i.e. arrows, is taken to be *associative*, i.e.

$$\begin{gathered} a \xrightarrow{\ f\ } b \xrightarrow{\ g\ } c \xrightarrow{\ h\ } d \\ h \circ (g \circ f) = (h \circ g) \circ f. \end{gathered} \tag{3-46}$$

Of course this definition is restricted to the case where composition can be performed, that is when co-domains and domains are compatible.

A *category* is the restriction of a metacategory to the case where the objects are sets. A *graph*, also called diagram scheme, is a set O of objects and a set A of arrows (morphisms) and two functions

$$A \underset{\text{codom}}{\overset{\text{dom}}{\rightrightarrows}} O. \tag{3-47}$$

The arrows which can be composed are elements of the set of ordered pairs

$$A \times_O A = \{(g, f) \mid g, f \in A,\ \text{and}\ \mathrm{dom}\, g = \mathrm{codom}\, f\}. \tag{3-48}$$

This is the product over O. A *category* is a graph having the two additional functions

$$\begin{aligned} O &\xrightarrow{\quad \mathrm{Id} \quad} A, & \qquad\qquad A \times_O A &\xrightarrow{\quad \circ \quad} A, \\ c &\xrightarrow{\qquad} \mathrm{Id}_c, & g \times f &\xrightarrow{\qquad} h = g \circ f, \end{aligned} \tag{3-49}$$

with compatible domains and co-domains and assuming associativity. A further notion in the category C is

$$\mathrm{Hom}(b,c) \;=\; \{f \mid f \text{ in } C,\ \text{dom}\, f = b,\ \text{codom}\, f = c\} \tag{3-50}$$

which is equivalent to the set of arrows of C. A generalization of these notions is possible, see Oziewicz [99], where one finds n-categories, sketches and operads related to ideas we are using here. The idea is to study graphs like

$$\ldots \quad \rightrightarrows B \rightrightarrows A \underset{\text{codom}}{\overset{\text{dom}}{\rightrightarrows}} O \tag{3-51}$$

where one examines morphisms of arrows and morphisms of morphisms of ... of arrows. In fact, it is possible to define a category consisting only of arrows and morphisms of arrows.

We have to remark, that the sets used in categories here will be so called *small sets* which are tame sets in the sense of set theory, i.e. one disallows pathological sets as the 'set of all sets' etc. to avoid antinomies. Such categories are called *small categories*.

Examples of categories are among the following:

0 the empty category, no objects no arrows.

1 the identity category, one object, one (identity) arrow.

Set *Set:* Objects are small sets, arrows are functions between them.

Mon *Monoid:* A monoid can be addressed itself as a category with one object and arrows, among them the identity arrow. The category **Mon** is that where the objects are small monoids and the arrows are morphisms of monoids.

$\mathbb{k}$-**Mod** $\mathbb{k}$-*Modules:* small modules over the commutative ring $\mathbb{k}$.

A-**Mod** A-*modules:* small left A-modules and morphisms of modules.

Grp *Groups:* Objects are small groups, arrows are morphisms of groups. A group itself is a category with one object where all morphisms are isomorphisms.

Top *Topological spaces:* small topological spaces and continuous maps.

A *functor* is a morphism of categories. A functor consists therefore of two morphisms, a morphism of objects and a morphism of arrows. This opens two possibilities, either the directions of the arrows before and after the transformation are 'parallel', i.e. the direction is not changed, such functors are called covariant. Otherwise the functor reverses the direction of arrows and is called contravariant. We will not have much opportunity to enter this topic, but the notion of opposite algebras, products etc. touches this fact.

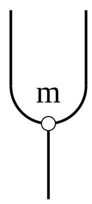
Some more notation is appropriate to make contact with the current literature. If an arrow $f\ :\ a \to b$ in a category **C** is invertible, i.e. there exists $f^{-1}\ :\ b \to a$ with $f \circ f^{-1} = \mathrm{Id} = f^{-1} \circ f$

including the domain and codomain restriction, then a and b are *isomorphic* written as $a \cong b$. An arrow $m \ : \ a \rightarrow b$ is called *monic*, and the corresponding morphism is a *monomorphism* in **C**, if for any two parallel arrows $f, g \ : \ d \rightarrow a$ the equality $f \circ m = g \circ m$ implies $f = g$. That is there holds a right cancellation law. An arrow h is called *epi*, and its corresponding morphism is an *epimorphism* if there holds analogously a left cancellation law. In **Set** and in **Grp** monic arrows, i.e. monomorphisms, are injections and epi arrows, i.e. epimorphisms, are surjections.

We will have below occasion to see that graphical calculus is dangerous in that sense that it is difficult to keep track of the epi- and mono morphism properties of arrows. The discussion of Kuperberg's Lemma 3.2 will show how graphical calculus can be misleading. One pays a price for a nice representation with a certain peculiarity in calculating with them.

Note, that in a commutative diagram as given in Eqn. 3-45 the objects are localized and the morphisms are given as arrows. This goes directly with intuition where one expects objects to be 'solid' or 'material' as indicated by the name and arrows are seen to by 'dynamical', 'operating' or 'transforming' entities. That this is not a necessity can be shown directly by categories where objects are themselves morphisms. One is therefore able to develop a graphical notation which reverses this assumption and depicts the objects as lines and the morphisms as points on this lines or as other localized operations as connecting or splitting lines. Such diagrams will be called *tangles* [139, 89, 88]. As an example we give the notion of a product in both representations

$$\begin{array}{c} A \otimes A \\ \Big\downarrow m \\ A \end{array}$$



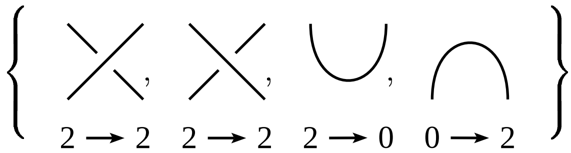
(3-52)

The tangle is read downwards unless arrows are used to indicate which lines have to be traversed in which direction. As commutative diagrams can be read for elements or for sets of objects, or even for categories themselves, i.e. diagrams which contain functors, this is true also for tangle relations. Actually a tangle can be seen as constituting a dual type of graph. We think that tangles can be intuitively understood as a *process*. A set of objects can move along the line of a tangle suffering certain operations (morphisms). This is much more a dynamical picture closely related to physical processes where also objects like point particles or quanta 'move' around subjected to forces or quantum processes such as creation or annihilation. The multiplication shown above can be seen as the annihilation of two factors into a newly created entity, the product, of possibly different type.

The notion of a category such allows to speak about Hopf objects, which are elements of the objects of the category of Hopf algebras etc.

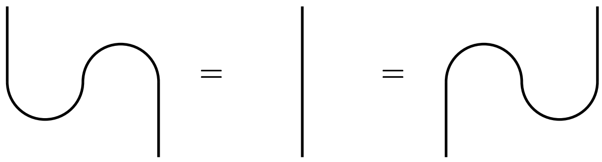
3.2.2 Tangles for knot theory

As an example we examine tangles of a special kind arising from knot theory and link invariants, see e.g. [99]. The projection of 3-dimensional knots into a plane, yields a planar graph which contains information about the knot in three space. If additionally the information is conserved which string of the knot crosses over and which under the other w.r.t. the particular projection, the planar graph contains all topological information about the knot. A classification of such planar graphs would then classify the knots. We can state an alphabet, i.e. a basic set, for (graphs of) knots and links in the following way:



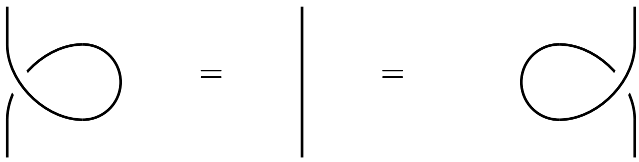
(3-53)

Knots and links are composed from these basic words so that no loose ends are open. However, since a knot can have different planar projections, one has to introduce rules which allow to transform one such representation into another. In this case, these are the Reidemeister moves. The first assertion is:



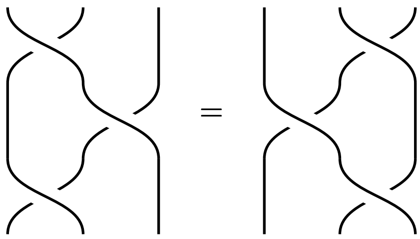
(3-54)

which states that lines can be bent at will like rubber strings. The second move is



(3-55)

and an analogous tangle equation for the second crossing, stating that a single loop can be untwisted. The last move is a braid relation and shows how to move a single line in the middle of two other lines through a crossing:



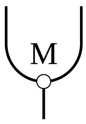
(3-56)

This is in fact the braid relation of an Artin braid group which is generated by the crossings at the position $i \in \{1, \ldots, n-1\}$, if n-strings are given. If one adds tangles, projections of knots or links and to multiplies them by numbers, the set of basic tangles as given in Eqn. 3-53 and composed so that no outgoing lines occur, then constitutes a basis of an infinite dimensional free module. The relations asserted by the Reidemeister moves lead to equivalence classes of tangles which also constitute a basis of a module. If relations between the letters of the basic alphabet hold, such relations are called *skein relations.* A module having skein relations on its constituting alphabet is a *skein module*.

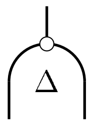
3.2.3 Tangles for convolution

The convolution alphabet [99] is given by a multiplication map which is of type $2 \to 1$, and a dual structure called *co-product*. The co-product arrises from the product by categorial duality. Product and co-product form together the convolution alphabet.

$$\begin{tikzcd} A \otimes A \arrow[d, "M"] \\ A \end{tikzcd} \cong$$



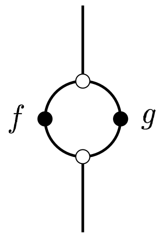
$$\begin{tikzcd} A \arrow[d, "\Delta"] \\ A \otimes A \end{tikzcd} \cong$$



(3-57)

The convolution product is defined in the following way using either commutative diagrams or tangles:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\star"'] & A \otimes A \arrow[d, "f \otimes g"] \\ A & A \otimes A \arrow[l, "m"'] \end{tikzcd}$$



(3-58)

The convolution product turns the module of endomorphisms into an algebra, the *convolution algebra*. In terms of algebraic symbolism we can write down this process as

$$(f \star g)(x) = m \circ (f \otimes g) \circ \Delta(x) \;=\; \sum_{(x)} f(x_{(1)})g(x_{(2)}). \tag{3-59}$$

The element x is from the objects of a category, f, g are morphisms. The convolution product $\star$ is therefore from the morphisms of morphisms and we would enter here the case of a 2-category, but see Oziewicz for the general case and proper definitions [99].

The commutative diagrams and tangles of Hopf algebras and Hopf gebras will be given below where these objects will be defined. However, we have seen some of them in the Kuperberg notation already, see Eqns. 3-31 to 3-35. Products and co-products have been displayed already in Eqn. 3-57, while the convolution was given in Eqn. 3-58.

We should finally remark that the Kuperberg graphical notation is a variant of the tangle notation since it localizes also the morphisms and depicts the objects by arrows. However, there are differences, and as we will see later, the notion of duality etc. is not so clearly expressed in Kuperberg's representation.

Chapter 4

Hopf algebras

Hopf algebras were introduced by Heinz Hopf [72] to study topological aspects of group manifolds and their generalizations. From this origin it is clear that Hopf algebras are closely related to *groups*, as we will see more clearly below, and *topology*. Indeed Hopf algebras play an important role in the theory of link invariants, knot theory and lattice models of various types. For our purpose this fact does not directly come into play, but should kept in mind.

The name *Hopf algebra* was, for obvious reasons, not coined by Hopf, but by Milnor and Moore [94]. The usage of this term by Borel [16] for a structure without the diagonalization (complication, diagonal map or co-product) has to be rejected. Moreover, Milnor and Moore's Hopf algebra comes nowadays under the name of a *bialgebra* and only antipodal, i.e. group like see below, such structures are named *Hopf algebras* today. Detailed descriptions of Hopf algebras may be found e.g. in Milnor and Moore [94], the standard reference by Sweedler [130] and Abe [1]. Our presentation is along the trail of Milnor and Moore.

We will later on contrast the notion of a *Hopf al-gebra* with that of a *Hopf gebra* which we will derive from a *convolution algebra* following Oziewicz [102, 103]. This approach seems to be mathematically more sound and provides us a better classification and understanding of various *al*- and *co*-gebraic structures including Hopf gebras. Moreover, following Bourbaki [18] the naming *gebra* is linguistically correct and does not abuse its Arabic origin[1].

Before we start to resume the mathematical details needed for our treatise, we give a genealogy of al-gebras and co-algebras to display the further development in a diagram:

[1]The name algebra goes back to a book of Abu Ja'far Mohammad ibn Mussa al Khwārizmi, (780–850), see [132, 98], named *Al-ǰabr w'al muqābala*, Baghdad, Iraq, where for the first time algebraic methods have been systematically developed. From 'al-ǰabr' , transcribed as 'the completion' the word algebra is derived. In German transcriptions one finds even a ğ in place of the English ǰ. Unfortunately the article *al* was incorporated. The Bourbaki group had already suggested to use therefore al-gebra, but co-gebra, bi-gebra, Hopf gebra etc. which comes to meet the usage of the Arabic language without abusing it. From a further book of Kwārizmi available only in a Latin translation, *Algorithmi de numero indorum*, i.e. *Al Khwārizmi on the Hindu Art of Reckoning*, the term *algorithm* was taken.

$$\begin{tikzcd} \text{Al-Gebra} \arrow[rr, leftrightarrow, "\text{duality}", "\text{Mutual Compatibility}"'] \arrow[dr] & & \text{Co-Algebra} \arrow[dl] \\ & \text{Bi-Algebra} \arrow[d, "\exists\ \text{Antipode}"] & \\ & \text{Hopf Algebra} & \end{tikzcd}$$

Algebras and *Co-algebras* are mutually dual structures in a certain sense as we will explain below. If certain compatibility laws hold, the selfdual structure is called a bi-algebra. If in addition an antipode exists, one obtains a Hopf algebra.

4.1 Algebras

4.1.1 Definitions

Let $\Bbbk$ be a commutative ring or even a field which is chosen once and fixed thereafter. We consider finitely generated $\Bbbk$-modules, denoted by capital letters $A, B, C, \ldots, V, W, \ldots$. Tensor products of two $\Bbbk$-modules are taken over $\Bbbk$ and will be denoted as $A \otimes_{\Bbbk} B \equiv A \otimes B$. These $\Bbbk$-modules constitute a category. $\hom(A, B)$ denotes the morphisms of A into B in this category, while $A, B, C, \ldots$ are the objects.

A *graded* $\Bbbk$-module is a (finite) family of $\Bbbk$-modules $\{A_n\}$ where n runs through the non-negative integers. n is called the *degree*. If A, B are graded $\Bbbk$-modules, a graded morphism of graded $\Bbbk$-modules $f : A \to B$ is a family of morphisms $\{f_n\}$ such that $f_n : A_n \to B_n$ is a morphism of $\Bbbk$-modules.

If A and B are graded $\Bbbk$-modules, then $A\hat{\otimes}B$ is a graded $\Bbbk$-module such that $(A\hat{\otimes}B)_n = \oplus_{i+j=n} A_i\hat{\otimes}B_j$. Commonly the *hat* in $\hat{\otimes}$ is dropped if the grading is clear. The graded tensor product is a particular case of a crossed product, see below. If $f : A \to A'$ and $g : B \to B'$ are graded morphisms of graded $\Bbbk$-modules then $(f\hat{\otimes}g) : A \hat{\otimes} B \to A' \hat{\otimes} B'$ is the graded morphism of graded $\Bbbk$-modules such that $(f\hat{\otimes}g)_n = \oplus_{i+j=n} f_i\hat{\otimes}g_j$.

If A is a graded $\Bbbk$-module, we denote by A^* the graded $\Bbbk$-module such that $A^*_n = \hom(A_n, \Bbbk)$. A^* is called the dual $\Bbbk$-module, elements of A^* are forms. If these forms are linear and $\Bbbk$ a field, the elements of A^* are called co-vectors. Morphisms of graded dual $\Bbbk$-modules are defined as in the case of graded $\Bbbk$-modules. The identity morphism will be denoted by Id or eventually by the same symbol as for the $\Bbbk$-module:

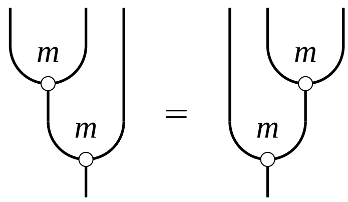
$$A \xrightarrow{\quad A \quad} A \quad \equiv \quad A \xrightarrow{\quad \mathrm{Id} \quad} A. \tag{4-1}$$

If necessary, $\Bbbk$ will be considered as graded $\Bbbk$-module which is defined to be the 0-module in all degrees except 0, and the ring (field) $\Bbbk$ in degree 0. This definition is equivalent to $A \hat{\otimes} \Bbbk = A = \Bbbk \hat{\otimes} A$, where A and $\Bbbk$ are (graded) $\Bbbk$-modules.

The notion of an algebra emerges directly from the writings of H. Graßmann [64, 63]. Graßmann denotes any *bilinear*, that is left and right distributive, map a *multiplication*, hence allowing non-associative multiplications, but keeping linearity. In our treatise, we will be interested in linear and associative multiplications most of the time and will explicitly state when these assumptions are not met. Hopf algebras are usually defined, however, as associative, but linearity of the multiplication map may not be assumed.

Definiton 4.1. *A* graded unital algebra *over $\Bbbk$ is a graded $\Bbbk$-module A together with morphisms of graded $\Bbbk$-modules $m \;:\; A\hat{\otimes}A \to A$ (multiplication) and $\eta \;:\; \Bbbk \to A$ (unit) such that the diagrams*

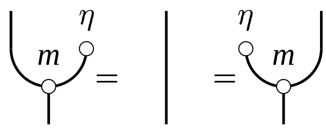
$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mathrm{Id}\otimes m"] \arrow[d, "m\otimes \mathrm{Id}"] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$



(4-2)

and

$$\begin{tikzcd} & A\otimes A \arrow[d, "m"] & \\ A\otimes \Bbbk \arrow[ur, "A\otimes\eta"] & A \arrow[l, "\approx"'] \arrow[r, "\approx"] & \Bbbk\otimes A \arrow[ul, "\eta\otimes A"'] \end{tikzcd}$$



(4-3)

are commutative.

If the multiplication m has no unit, we speak about a non-unital algebra. A prominent example of such algebras are the Lie algebras, which however fail to be associative hence they are not algebras in the above defined sense.

Definiton 4.2. *The* graded switch *(crossing, scattering) $\hat{\tau} \;:\; A\,\hat{\otimes}\,B \to A\,\hat{\otimes}\,B$ is a morphism of graded $\Bbbk$-modules A and B such that $\hat{\tau}_n(a\,\hat{\otimes}\,b) = (-1)^{\partial a \partial b}\; b\,\hat{\otimes}\,a$ for $a \in A_p$, $b \in B_q$ and $p+q=n$. The degree of a homogenous element c is denoted by ∂c.*

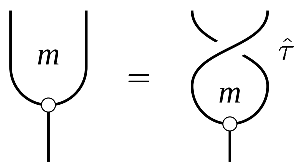
As a tangle the crossing reads:

(4-4)

If the grading is trivial, i.e. $A = A_0$ and $B = B_0$, one obtains through this definition the usual switch, as used e.g. by Sweedler [130]

Definiton 4.3. *A graded unital algebra is called* commutative *if the diagram*

$$\begin{tikzcd} A \otimes A \arrow[dd, "\hat{\tau}"] \arrow[dr, "m"] & \\ & A \\ A \otimes A \arrow[ur, "m"'] & \end{tikzcd}$$



(4-5)

is commutative (the tangle identity holds).

We follow the convention to talk about commutativity if the standard crossing is employed in the above tangle, the precise term would however be 'graded commutative'.

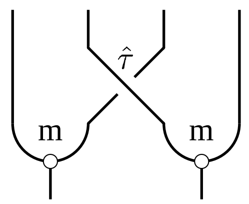
Note: If the graded $\Bbbk$-module A is of the form $A = \Bbbk \oplus A_1$ such algebras are called usually anticommutative. If one finds $A = A_0 \oplus A_1 = (\Bbbk \oplus A_0') \oplus A_1$ one deals with a supersymmetric algebra, which obeys a $\mathbb{Z}_2$-grading. In the tangle diagram one sees that in a commutative algebra the crossing is absorbed by the product morphism m

$$m = m \circ \hat{\tau}. \tag{4-6}$$

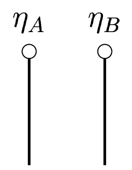
Definiton 4.4 (Crossed product). *If A and B are graded unital algebras over $\Bbbk$, $A \hat{\otimes} B$ is turned into an algebra with multiplication M and unit η via*

$$\begin{aligned} M &: A\hat{\otimes}B \times A\hat{\otimes}B \to A\hat{\otimes}B \\ M &= (m_A \otimes m_B) \circ (\mathrm{Id} \otimes \hat{\tau} \otimes \mathrm{Id}) \qquad (4\text{-}7) \\ \eta &: \Bbbk = \Bbbk \otimes \Bbbk \to A \hat{\otimes} B \\ \eta &= \eta_A \otimes \eta_B. \qquad (4\text{-}8) \end{aligned}$$

In terms of tangles this reads:



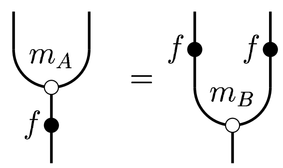
and



(4-9)

A morphism of algebras $f : A \to B$ is a morphism of graded $\Bbbk$-modules which fulfils

$$\begin{tikzcd} A \otimes A \arrow[r, "m_A"] \arrow[d, "f \otimes f"] & A \arrow[d, "f"] \\ B \otimes B \arrow[r, "m_B"] & B \end{tikzcd}$$



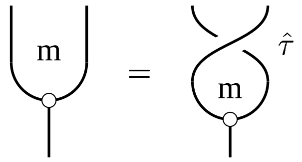
(4-10)

In other words, the multiplications in A and B are related via

$$f \circ m_A = m_B \circ (f \otimes f). \tag{4-11}$$

Furthermore, an algebra A is commutative –in the sense defined above– if and only if the product $m_A \ : \ A \otimes A \to A$ is a morphism of algebras

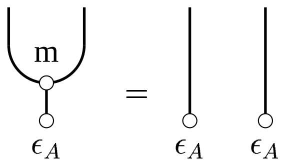
$$\begin{tikzcd} (A\otimes A)\otimes(A\otimes A) \arrow[r, "m_A\otimes m_A"] \arrow[d, "m_A\otimes m_A"] & A\otimes A \arrow[d, "m_A"] \\ A\otimes A \arrow[r, "m_A"] & A \end{tikzcd} \quad \Rightarrow \tag{4-12}$$



where the conclusion follows by a simple calculation using associativity. The crossed product is in a certain sense a canonical generalization of the product m_A on the algebra $A \otimes A$.

The *augmentation* of an algebra A is a morphism of algebras $\epsilon \ : \ A \to \Bbbk$

$$\begin{tikzcd} A\otimes A \arrow[r, "m_A"] \arrow[d, "\epsilon\otimes\epsilon"] & A \arrow[d, "\epsilon"] \\ \Bbbk\otimes\Bbbk \arrow[r, "m_{\Bbbk}"] & \Bbbk \end{tikzcd} \tag{4-13}$$



Obviously ϵ_A is a (linear) form on the algebra. An algebra A together with an augmentation ϵ_A is called an augmented (or supplemented) algebra. If A is an augmented algebra, we denote by $I(A)$ the kernel of $\epsilon_A \ : \ A \to \Bbbk$. Clearly one has $I(A)_q = A_q$ for $q > 0$ and $I(A)_0$ is the kernel of $\epsilon_0 \ : \ A_0 \to \Bbbk$. The ideal $I(A)$ is called the augmentation ideal of A. As a graded $\Bbbk$-module A may be decomposed as the direct sum

$$A = \mathrm{img}\,\eta \oplus \ker \epsilon \tag{4-14}$$

or identifying $\Bbbk = \mathrm{img}\,\eta$

$$A = \Bbbk \oplus \ker \epsilon = \Bbbk \oplus I(A), \tag{4-15}$$

where the fact was used that $\epsilon \circ \eta \ : \ \Bbbk \to \Bbbk$ is the identity map on $\Bbbk$.

Note: The fact that $I(A)$ is an ideal in A, is directly related to the property that the augmentation form ϵ is an algebra homomorphism. One may note the similarity between the real and imaginary parts of complex, quaternionic and octonionic division algebras. An augmented algebra generalizes such a structure. We can therefore loosely speak about the real part $\mathrm{img}\,\eta = \Bbbk$ and the imaginary part $\ker \epsilon = I(A)$ for connected such algebras.

Definiton 4.5. *An algebra over* $\Bbbk$ *is* connected *if* $\eta \ : \ \Bbbk \to A_0$ *is an isomorphism.*

In that case the algebra has a unique augmentation $\epsilon \ : \ A \to \Bbbk$ such that $\Bbbk \cong A_0$ and $\Bbbk \xrightarrow{\eta} A_0 \xrightarrow{\epsilon} \Bbbk$ where $\epsilon_0\eta_0 = \mathrm{Id}_{\Bbbk}$.

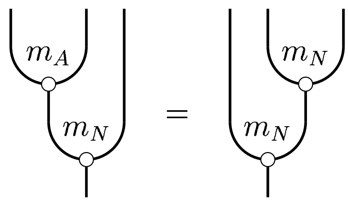
4.1.2 A-modules

A-modules are needed to study the representation theory of algebras also group algebras, and of various groups connected with them. From a mathematical point of view every module is well suited for this purpose as long as an action can be defined on it. We will, however, be mostly interested in representations of the multiplicative (semi) group on the module the algebra is built over. Concerning physics, there is no reason to introduce a new entity (remember that 'particles' are defined as elements of irreducible modules), i.e. a new type of module, which is foreign to the problem at hand. Moreover, for our cases we find all fundamental irreducible representations as (sub)spaces of the module A and can build up any representation from them.

To distinguish A-modules and algebras, we use letters $N, M, \ldots$ for A-modules and do not assume here that N is a submodule of the algebra $N \subseteq A$.

Definiton 4.6. *If A is a graded unital algebra over $\mathbb{k}$, a graded left A-module is a graded $\mathbb{k}$-module N together with a morphism $m_N : A \otimes N \to N$ such that the diagrams*

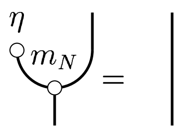
$$\begin{tikzcd} A \otimes A \otimes N \arrow[r, "\mathrm{Id} \otimes m_N"] \arrow[d, "m_A \otimes \mathrm{Id}"] & A \otimes N \arrow[d, "m_N"] \\ A \otimes N \arrow[r, "m_N"] & N \end{tikzcd}$$



(4-16)

and

$$\begin{tikzcd} & A \otimes N \arrow[d, "m_N"] \\ \mathbb{k} \otimes N \arrow[ur, "\eta \otimes \mathrm{Id}"] & N \arrow[l, "\approx"'] \end{tikzcd}$$

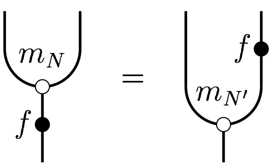


(4-17)

are commutative.

A morphism of graded A-modules $f : N \to N'$ is a morphism of graded $\mathbb{k}$-modules N, N' such that the diagrams

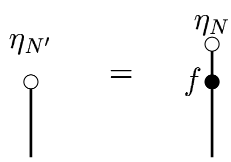
$$\begin{tikzcd} A \otimes N \arrow[r, "m_N"] \arrow[d, "\mathrm{Id} \otimes f"] & N \arrow[d, "f"] \\ A \otimes N' \arrow[r, "m_{N'}"] & N' \end{tikzcd}$$



(4-18)

and

$$\begin{tikzcd} \mathbb{k} \arrow[r, "\eta_N"] \arrow[d, "\mathrm{Id}_{\mathbb{k}}"] & N \arrow[d, "f"] \\ \mathbb{k} \arrow[r, "\eta_{N'}"] & N' \end{tikzcd}$$



(4-19)

are commutative. Compare this relation with that of Eqn. 4-10. Morphisms of A-modules can be added $(f+g) \,:\, N \rightarrow N'$ via adding the particular degrees of f, g. Kernels and cokernels are defined as $(\ker f)_q = \ker f_q$ etc.

Right A-modules are defined via the right action in a similar way.

Analogously to algebras we define connected modules as a $\Bbbk$-module if $N_0 \cong \Bbbk$. We will see later on that Graßmann algebras are connected and have connected modules, while Clifford algebras are not connected.

4.2 Co-algebras

4.2.1 Definitions

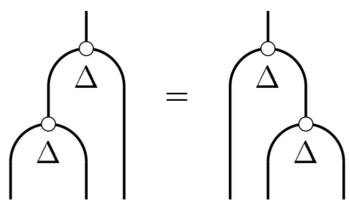
We use now categorial duality, i.e. the reversion of arrows in commutative diagrams or the horizontal mirroring of tangles, to define a dual structure called co-algebra.

Definiton 4.7. *An unital* co-algebra *over $\Bbbk$ is a graded $\Bbbk$-module C together with morphisms Δ, the co-multiplication, and ϵ, the counit, of graded C-modules*

$$\begin{aligned} \Delta &: C \rightarrow C \otimes C \\ \epsilon &: C \rightarrow \Bbbk \end{aligned} \tag{4-20}$$

such that the diagrams

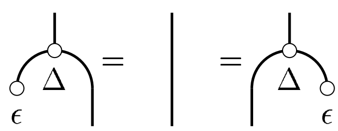
$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{Id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{Id}"] & C \otimes C \otimes C \end{tikzcd}$$



(4-21)

and

$$\begin{tikzcd} \Bbbk \otimes C \arrow[dr, "\approx"'] & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{Id}"'] \arrow[r, "\mathrm{Id} \otimes \epsilon"] & C \otimes \Bbbk \arrow[dl, "\approx"] \\ & C \arrow[u, "\Delta"] & \end{tikzcd}$$

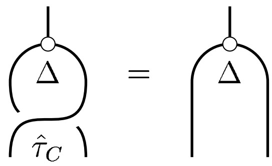


(4-22)

are commutative.

The co-algebra is called co-commutative (or simply commutative in a unique context) if the diagram

$$\begin{tikzcd} C \otimes C \arrow[dd, "\hat{\tau}_C"'] & \\ & C \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ C \otimes C & \end{tikzcd}$$



(4-23)

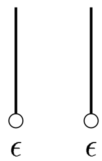
is commutative.

A *crossed co-product* of co-algebras C and D over $\mathbb{k}$ is defined as

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{Id} \otimes \hat{\tau} \otimes \mathrm{Id}} (C \otimes D) \otimes (C \otimes D) \tag{4-24}$$

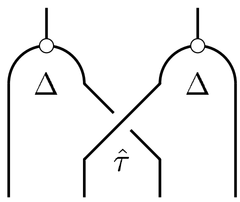
where the co unit is defined as

$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathbb{k} \otimes \mathbb{k} \approx \mathbb{k}.$$



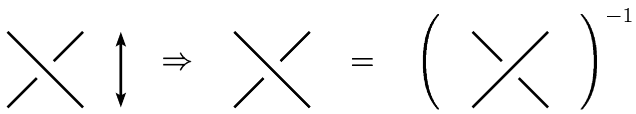
(4-25)

In terms of tangle one displays the crossed co-product as:



(4-26)

Note: Observe that in older literature e.g. Sweedler [130] and Milnor and Moore [94] it is assumed that the crossing $\hat{\tau}$ is an involution, i.e. that $\hat{\tau}^2 = \mathrm{Id} \otimes \mathrm{Id}$ hence $\hat{\tau}^{-1} = \hat{\tau}$. If we do not impose this restriction, it could be convenient to introduce the crossed co-product using the inverse crossing $\hat{\tau}^{-1}$ in spite of the fact that mirroring the tangle

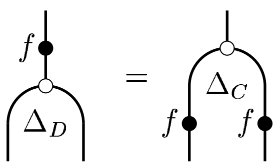


(4-27)

yields the crossing back, assuming that the over and under information is corerctly encoded.

A morphism of co-algebras $f : C \to D$ is a morphism of graded $\mathbb{k}$-modules such that the diagrams

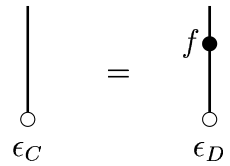
$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "f"] & C \otimes C \arrow[d, "f \otimes f"] \\ D \arrow[r, "\Delta_D"] & D \otimes D \end{tikzcd}$$



(4-28)

and

$$\begin{tikzcd} C \arrow[r, "\epsilon_C"] \arrow[d, "f"] & \mathbb{k} \arrow[d, "\mathrm{Id}_{\mathbb{k}}"] \\ D \arrow[r, "\epsilon_D"] & \mathbb{k} \end{tikzcd}$$



(4-29)

are commutative. A co-algebra is co-commutative if and only if $\Delta : C \to C \otimes C$ is a morphism of co-algebras. The proof follows by duality from the proof for the case of algebras.

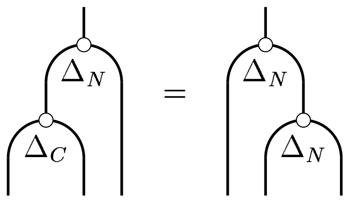
One can consider $\mathbb{k}$ to be a co-algebra in a canonical way. This allows to introduce an *augmentation* of co-algebras as a morphism of co-algebras $\eta : \mathbb{k} \to C$. If C is an augmented co-algebra, i.e. a co-algebra with augmentation η, we denote by $J(A)$ the cokernel of η. Considering C as a $\mathbb{k}$-module we find

$$C = \mathbb{k} \oplus J(A). \tag{4-30}$$

4.2.2 C-comodules

Let C be a co-algebra over $\mathbb{k}$. A left C-comodule is a graded $\mathbb{k}$-module N together with morphisms $\Delta_N : N \to C \otimes N$ such that the diagrams

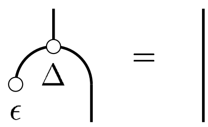
$$\begin{tikzcd} N \arrow[r, "\Delta_N"] \arrow[d, "\Delta_N"'] & C \otimes N \arrow[d, "\mathrm{Id} \otimes \Delta_N"] \\ C \otimes N \arrow[r, "\Delta_C \otimes \mathrm{Id}"] & C \otimes C \otimes N \end{tikzcd}$$



(4-31)

and

$$\begin{tikzcd} \mathbb{k} \otimes C \arrow[dr, "\approx"'] & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{Id}"'] \\ & C \arrow[u, "\Delta"'] \end{tikzcd}$$

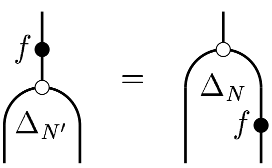


(4-32)

are commutative.

Let N, N' are left C-modules. A morphism $f : N \to N'$ of left C-modules is a morphism of graded $\mathbb{k}$-modules such that the diagram

$$\begin{tikzcd} N \arrow[r, "\Delta_N"] \arrow[d, "f"] & C \otimes N \arrow[d, "\mathrm{Id} \otimes f"] \\ N' \arrow[r, "\Delta_{N'}"] & C \otimes N' \end{tikzcd}$$



(4-33)

is commutative.
Note: Beside the completely parallel developments of algebras, co-algebras and modules, co-modules due to categorial duality, they exhibit, after a close look, some different features. As an example, one should note, that the left A-modules constitute an abelian category, while the left C-comodules do not unless further conditions are assumed [94]. Since we do not need such sophisticated facts here, the interested reader should consult the original literature.

4.3 Bialgebras

4.3.1 Definitions

A graded $\Bbbk$-module A is of *finite type* if each A_n is finitely generated as $\Bbbk$-module and only a finite number of A_n's are not zero. It is called projective if each A_n is projective.

Under these conditions one obtains that the morphism of $\Bbbk$-modules

$$\lambda\ :\ A \rightarrow A^{**} \tag{4-34}$$

defined by $\lambda(x)a^* = a^*(x)$ for $x \in A_n$, $a^* \in A^*_n$ is an isomorphism, and the morphism of graded $\Bbbk$-modules

$$\alpha\ :\ A^* \otimes B^* \rightarrow (A \otimes B)^* \tag{4-35}$$

defined as $\alpha(a^* \otimes b^*)(x \otimes y) = a^*(y)b^*(x)$ for $a^* \in A^*_p$, $b^* \in B^*_q$, $y \in A_p$, $x \in B_q$ is an isomorphism. That is the tensor product of linear forms is a linear form itself acting on a tensor product in a canonical way. This allows us to write $A^{**} = A$ and $A^* \otimes B^* = (A \otimes B)^*$. One could prove, that A^* is of projective finite type if A is of projective finite type and vice versa.

The product m_A from A induces now a *co-product* $m^*_{A^*}$ by categorial duality also abusing our notation. In other words, a product of vectors implies a co-product on co-vectors and a products on co-vectors implies a co-product on vectors by so called product co-product duality. Having a pair of a product and co-product on A and A^*, we find four products which might be independently chosen.

Theorem 4.8 (Duality). *Suppose that A is a graded $\Bbbk$-module of projective and finite type, then:*

(1) $m_A\ :\ A \otimes A \rightarrow A$ *is a multiplication in* A *if and only if* $m^*_{A^*}\ :\ A^* \rightarrow A^* \otimes A^*$ *is a co-multiplication in* A^*.

(2) m_A *is associative if and only if* $m^*_{A^*}$ *is co-associative.*

(3) $\eta\ :\ \Bbbk \rightarrow A$ *is a unit for the multiplication* m_A *if and only if* $\eta^*\ :\ A^* \rightarrow \Bbbk^* \cong \Bbbk$ *is a counit for the co-multiplication* $m^*_{A^*}$.

(4) (A, m_A, η) *is an associative unital algebra if and only if* $(A^*, m^*_{A^*}, \eta^*)$ *is a co-associative counital co-algebra.*

(5) $\epsilon : A \to \mathbb{k}$ *is an augmentation of the algebra* (A, m_A, η) *if and only if* $\epsilon^* : \mathbb{k} \to A^*$ *is an augmentation of the co-algebra* $(A^*, m^*_{A^*}, \eta^*)$.

(6) The algebra (A, m_A, η) *is commutative if and only if the co-algebra* $(A^*, m^*_{A^*}, \eta^*)$ *is co-commutative.*

The preceeding theorem exhibits a beautiful symmetry. Moreover, one notes, that if an algebra structure is employed on the module A^* of linear forms, this induces automatically an co-algebra structure on the (double) dual module $A^{**} = A$.

At this point one should reconsider the algebra of meet and join, i.e. the Graßmann-Cayley algebra. Since the join acts on *points* $\in A$ and the meet acts on *planes* $\in A^*$, the join induces a co-algebra structure $\Delta_\vee$ on the planes, i.e. on A^*, and the meet induces a co-algebra structure $\Delta_\wedge$ on points, i.e. on A. The duality between meet and join is mediated by Graßmann's Ergänzung, i.e. the orthogonal complement and the duality between products and co-products by categorial duality. This can be displayed in the diagram

$$\begin{tikzcd} GC(\wedge,\vee,*) \arrow[r, leftrightarrow, "\star"] \arrow[d, leftrightarrow] & GC(\Delta_\vee,\Delta_\wedge,*) \arrow[d, leftrightarrow] \\ H_\wedge(\wedge,\Delta_\vee,*) \arrow[r, leftrightarrow, "\star"] & H_\vee(\vee,\Delta_\wedge,*) \end{tikzcd} \tag{4-36}$$

Hence the Graßmann-Cayley algebra is built over a pair of spaces $A \oplus A^*$, both seen as algebras, and this structure is equivalent to a Graßmann-Hopf algebra, see below.

Now, it is clear that one is tempted to complete the structure to go over to a four-fold algebra, see [43]. In fact this is also done using the quantum double [44, 91]

$$H \oplus H^* \cong GC(\wedge, \Delta_\vee, \vee, \Delta_\wedge, *) \tag{4-37}$$

Note that the introduction of two independent products on A and A^* results in a completely *independent* structure $(\wedge, \Delta_\vee)$ on A and $(\vee, \Delta_\wedge)$ on A^*. Hence product and co-product on *one* space are not related at all. This will motivate later on the study of convolution algebras having deformed products and *independently* deformed co-products. Unfortunately, because of the canonical identification of A^* with A via the co-vector basis ε^i fulfilling $\varepsilon^i(e_j) = \delta^i_j$, hides the fact that one deals with two independent structures.

Definiton 4.9 (Bialgebra). *A* bialgebra *over* $\mathbb{k}$ *is a graded* $\mathbb{k}$*-module* B *together with morphisms of graded* $\mathbb{k}$*-modules*

$$\begin{aligned} m_B &: B \otimes B \to B, \qquad & \eta_B &: \mathbb{k} \to B \\ \Delta_B &: B \to B \otimes B, \qquad & \epsilon_B &: B \to \mathbb{k} \end{aligned} \tag{4-38}$$

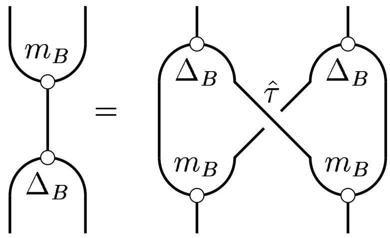
such that

(1) (B, m_B, η_B) *is an augmented* $\Bbbk$*-algebra,*

(2) $(B, \Delta_B, \epsilon_B)$ *is an augmented* $\Bbbk$*-co-algebra,*

(3) the diagram

$$\begin{tikzcd}
B \otimes B \arrow[r, "m_B"] \arrow[d, "\Delta_B \otimes \Delta_B"] & B \arrow[r, "\Delta_B"] & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "\mathrm{Id} \otimes \hat{\tau} \otimes \mathrm{Id}"] & & B \otimes B \otimes B \otimes B \arrow[u, "m_B \otimes m_B"']
\end{tikzcd}$$



(4-39)

is commutative.

Condition (3) states that Δ_B is an algebra homomorphism since it preserves also units and m_B is a co-algebra homomorphism preserving also counits. This follows from the fact that augmented algebras and co-algebras have been considered.

Note: This definition, even adopted in the vast majority of literature will be too narrow for our purpose. We will see that we have to drop the fact that B is an augmented algebra or co-algebra. In this case, Δ_B and m_B do not preserve the counit and unit respectively. Furthermore in older literature this structure is already called Hopf algebra, while we reserve this term for a still more restrictive setting.

Having defined the notion of a bialgebra, unnaturally emphasizing the algebra part, we can speak in a more symmetric fashion about bi-associativity if B is associative and co-associative, about B being bi-unital if it is unital and counital etc.

Using crossed products and crossed co-products one can establish a bialgebra action and coaction on the tensor product $N \otimes M$ of two left B-modules over $\Bbbk$. In tangle notation this reads:

(4-40)

where one has to use Δ_N and Δ_M in the r.h.s. tangle.

Milnor and Moore introduce a *quasi bialgebra* (quasi Hopf algebra in their notation) which does not assume associativity of multiplication and co-multiplication and where the augmentation is replaced by the condition

$$(4) \;\; \epsilon \circ \eta \;=\; \mathrm{Id}_{\Bbbk}, \qquad \Bbbk \xrightarrow{\epsilon \circ \eta} \Bbbk. \tag{4-41}$$

However, we are interested in associative multiplications exclusively and we will not follow this track.

If A is a graded $\Bbbk$-module, a *filtration* of A is a family $\{F_pA\}$ of sub-graded $\Bbbk$-modules of A, indexed by the integers such that $F_pA \subset F_{p+1}A$. The filtration $\{F_pA\}$ of the graded module A is *complete* if

$$\begin{aligned} &(1) \quad A = \lim_{p\to\infty} F_pA \\ \text{or} \qquad &(2) \quad A = \lim_{p\to-\infty} A/F_pA. \end{aligned} \tag{4-42}$$

A *filtered algebra* is an algebra over a graded $\Bbbk$-module such that the multiplication map $m_A : A \otimes A \to A$ is a morphism of filtered graded modules. A filtered left A-module M is a graded left A-module with filtration on its underlying graded $\Bbbk$-module such that the action $m_A : A \otimes M \to M$ is a morphism of filtered graded A-modules. One may easily generalize this notion to a filtered bialgebra.

If A is an augmented algebra over $\Bbbk$, let $Q(A) = \Bbbk \otimes_A I(A)$. The elements of the graded $\Bbbk$-module $Q(A)$ are called the *indecomposable elements* of A. If C is an augmented co-algebra over $\Bbbk$, let $P(C) = \Bbbk \Box_A J(A)$, where $\Box_A$ is the co-tensor product, see [94]. The elements of the graded $\Bbbk$-module $P(C)$ are called *primitive elements* of C. A bialgebra B is said to be *primitively generated* if the smallest sub-bialgebra of B containing $P(B)$ is B itself.

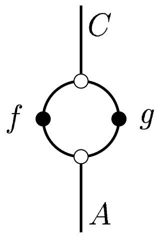
The notion of a filtration can now be used to generalize that of grading, connectedness and primitivity. However, while the augmentation was sufficient to prove the facts about the kernel $I(B)$ and the cokernel $J(B)$, i.e. determines the structure of $Q(B)$ and $P(B)$, this is no longer true for filtered (quasi) bialgebras. There one has to impose further conditions, that is to assume that certain exact sequences of the $Q(A)$ and $P(Q)$ modules are split, to be able to draw the conclusions. This fact will face us below, but we do not develop the corresponding theory since we have not yet applied it to the examples in physics.

4.4 Hopf algebras i.e. antipodal bialgebras

4.4.1 Morphisms of connected co-algebras and connected algebras : group like convolution

Let C be a connected co-algebra and A be a connected algebra, i.e. $\eta : \Bbbk \to A_0$ is an isomorphism, let $\mathrm{Conv}(C, A)$ be the set of morphisms $f : C \to A$ such that f_0 is the identity morphism of $\Bbbk$. If $f, g \in \mathrm{Conv}(C, A)$, the *convolution product* $f \star g$ is defined as the composition

$$C \xrightarrow{\Delta_C} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A \tag{4-43}$$



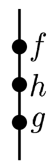
Theorem 4.10 (Milnor & Moore). *If C is a connected co-algebra and A is a connected algebra, then* $\mathrm{Conv}(C, A)$ *is a group under the convolution product $\star$ with identity (convolution unit)*

$$C \xrightarrow{\ \epsilon\ } \mathbb{k} \xrightarrow{\ \eta\ } A \tag{4-44}$$

Proof: $\mathrm{Conv}(C, A)$ is a monoid regarding its definition, and one needs to prove the existence of f^{-1} only. Suppose now that the action of f^{-1} is defined on degrees less than n, $x \in A_n$ and $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum'_{(x)} x_{(1)} \otimes x_{(2)}$, where the prime indicates a sum over proper cuts, i.e. $x_{(1)} \neq 1$ and $x_{(2)} \neq 1$. Assume $n > 0$ and recall that $0 <$ degree $x_{(2)} < n$ for proper cuts and all i. Let $f^{-1}(x) = -x - \sum'_{(x)} x_{(1)} f^{-1}(x_{(2)})$, which is the recursive definition of f^{-1}, since $f_0 \cong \mathrm{Id}_{\mathbb{k}}$. I.e. we have $f \star f^{-1} = \eta \circ \epsilon$.
Note that this recursive definition of f^{-1} is used, and was reinvented, by various authors. The most complete treatment might be found in Schmitt [126], where the antipode is constructed in this way, but also a non-recursive formula is given. The Connes-Kreimer antipode is, up to the renormalization scheme, calculated this way [33, 82, 25, 24, 23]. In fact f^{-1} is the inverse of Id and hence the antipode, but any other inverse can be obtained in this way.

If one considers morphisms $f : C \to C'$ of connected co-algebras and $g : A \to A'$ of connected algebras there is an induced morphism of groups $\mathrm{Conv}(f, g) : \mathrm{Conv}(C, A) \to \mathrm{Conv}(C', A')$

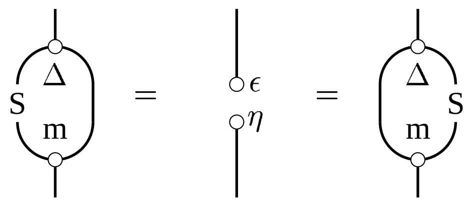
$$\mathrm{Conv}(f, g)h = ghf$$



(4-45)

Definiton 4.11 (Antipode). *Let B be a biconnected bialgebra, the* antipode *or* conjugation *of B is the (unique) inverse in* $\mathrm{Conv}(B, B)$ *of the identity morphism of B. The antipode is denoted as* S_B.

Let $U = \eta \circ \epsilon$ be the convolution unit as defined above. The defining relations of the antipode reads in tangle notation:

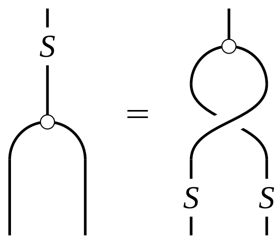


$$\mathrm{U} = \eta \circ \epsilon \ = \ \mathrm{Id}_{\mathbb{k}} \tag{4-46}$$

The following two theorems show that the antipode is intimately related with the notion of opposite products and opposite co-products. In fact, this establishes a further *duality* connecting the four spaces $H, H^{op}, H^{*}, H^{*op}$.

Theorem 4.12. *If B is a* biconnected bialgebra, *then the co-product diagram*

$$\begin{tikzcd} B \arrow[r, "\Delta"] \arrow[dd, "S"] & B \otimes B \arrow[d, "\hat{\tau}"] \\ & B \otimes B \arrow[d, "S \otimes S"] \\ B \arrow[r, "\Delta"] & B \otimes B \end{tikzcd}$$

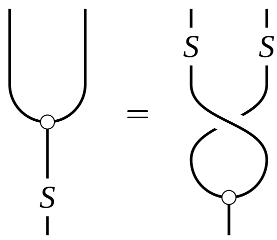


(4-47)

is commutative.

Theorem 4.13. *If B is a* biconnected bialgebra, *then the product diagram*

$$\begin{tikzcd} B \otimes B \arrow[r, "m"] \arrow[d, "S \otimes S"] & B \arrow[dd, "S"] \\ B \otimes B \arrow[d, "\hat{\tau}"] & \\ B \otimes B \arrow[r, "m"] & B \end{tikzcd}$$



(4-48)

is commutative.

Theorem 4.14. *If B is a biconnected bialgebra where either m_B or Δ_B is commutative, then $S \circ S \ :\ B \to B$ is the identity morphism of B, i.e. $S \circ S = \mathrm{Id}_B$ is an involution.*

The proofs will be discussed together with Kuperberg's Lemma 3.2, see below.

4.4.2 Hopf algebra definition

Definiton 4.15 (Hopf algebra, Milnor Moore). *A* Hopf algebra *is an antipodal biconnected bialgebra, i.e. a bialgebra which possesses an antipode.*

In fact, this raises a question if any antipodal bialgebra is a biconnected bialgebra, see also the discussion in section 6 of Fauser and Oziewicz [59]. We will later coin the term 'Hopf gebra' as used by Oziewicz which will not imply connectedness. In fact, the antipode definition etc. does not depend on connectedness.

The importance of these definitions for topology comes from the following: Consider the category of augmented co-algebras with (graded) commutative co-multiplication $\mathbf{Cog}(\eta, \hat{\tau})$. This category carries a product just by taking the tensor product $A \otimes B$ which needs essentially the commutativity of the co-multiplication. If $\Bbbk$ is a point in this category, one has the morphisms $\eta \ :\ \Bbbk \to C$, $\epsilon \ :\ C \to \Bbbk$ which turn the category $\mathbf{Cog}(\eta, \hat{\tau}, \epsilon \circ \eta, \otimes)$ with product into a monoid. We have thus seen above, that connected bialgebras with commutative co-multiplication are groups in the category $\mathbf{Cog}(\eta, \hat{\tau}, \epsilon \circ \eta, \otimes) = \mathbf{Hopf}$, i.e. Hopf algebras with involutive antipode.

If now $\mathbf{Top}_*$ is the category of topological spaces with base point, then if $\Bbbk$ is a field, there is a natural functor $H_*(\ \ ,\Bbbk)\ :\ \mathbf{Top}_* \to \mathbf{Hopf}$ which to every space X assigns its singular homology with coefficients in $\Bbbk$. The co-multiplication $H_*(X,\Bbbk) \to H_*(X,\Bbbk) \otimes H_*(X,\Bbbk)$ is the morphism induced by the diagonal map $\Delta\ :\ X \to X \otimes X$. This was the starting point of Hopf [72] and motivated the works of Milnor and Moore [94], Kuperberg [84, 85] and others.

Further notions like integrals will be defined below where they are explored in some examples.

Chapter 5

Hopf gebras

In this chapter we will develop the theory of *Hopf gebras* as opposed to that of *Hopf al-gebras* which was developed in the preceeding chapter. To some extend, Hopf gebras and Hopf al-gebras are equivalent, but it will turn out, that the notion of Hopf gebra allows a much clearer genealogy of al- and co-gebraic structures.

As we saw at various places, the Hopf algebras over a graded A-module are defined by the following structure (tensors): the associative multiplication $m_A : A \otimes A \to A$, the unit $\eta : \Bbbk \to A$ with $m(x \otimes \eta(1)) = x = m(\eta(1) \otimes x)\ \forall x \in A$, the associative co-multiplication $\Delta_A : A \to A \otimes A$, the counit $\epsilon : A \to \Bbbk$ with $(\mathrm{Id} \otimes \epsilon) \circ \Delta_A = \mathrm{Id} = (\epsilon \otimes \mathrm{Id}) \circ \Delta_A$, the antipode $S : A \to A$, an antihomomorphism and finally the crossing $\hat{\tau}$. One can summarize this as $H(A, m_A, \Delta_A, \eta, \epsilon; S; \hat{\tau})$.

The question which will be the guiding principle in this section is: Are the structure tensors independent? Already the presentation of Hopf algebras in the last chapter showed, that topological requirements as connectedness or the splitting of certain exact sequences played an important role to be able to show the existence of inverses which turned the convolution into a group.

The idea is not to start from algebras and co-algebras, but to take possibly non-unital and non-associative products and co-products to form a convolution algebra. Then it is a naturally given way to add structures unless one arrives at a Hopf gebra. We will restrict this general setting by assuming bi-associativity, i.e. associative products and co-associative co-products. Furthermore, we assume here that the product and co-product are endomorphic, so that source and target are the same $\Bbbk$-module A.

The fact that an antipodal convolution is already a Hopf gebra, follows from the theorem on the crossing derived by Oziewicz [102, 104], see below. One finds that an antipodal convolution has a unique crossing derived from the antipode since the antipode is unique. This idea will be generalized in the next chapter using convolutive idempotents.

In [59] co-convolutions have been introduced. It is clear that a convolution algebra is turned by categorial duality into a co-convolution co-algebra, however, we will not develop a theory of co-convolutions here.

A map of our further development might help to see how we proceed to obtain Hopf gebras.

$$
\begin{tikzcd}
\text{al-gebra, } (A, m_A) \arrow[rr, leftrightarrow, "\text{categorial duality}"] \arrow[dr] & & \text{co-gebra, } (C, \Delta_C) \arrow[dl] \\
 & \text{convolution algebra, } \mathrm{Conv}(C, A) \arrow[dl, "\exists U"'] \arrow[d, "\nexists U,\ \exists \hat{\tau}"] \arrow[dr, "\nexists U,\ \nexists \hat{\tau}"] & \\
\text{unital convolution} \arrow[d, "\exists S"'] \arrow[dr, "\nexists S,\ \exists \hat{\tau}"] \arrow[drr, "\nexists S,\ \nexists \hat{\tau}"] & \text{non-unital bigebra} & \text{non-unital convolution} \\
\text{antipodal Convolution i.e. Hopf gebra} & \text{unital bigebra} & \text{non-antipodal convolution}
\end{tikzcd}
\tag{5-1}
$$

We will derive some facts about the leftmost trail down to the Hopf gebras. There will not be an opportunity to go into the details of the other structures which occur in the above displayed diagram. The classification proposed here is different to that of the previous chapter. A bi-associative bi-connected Hopf gebra would fulfil the axioms of a Hopf algebra. We use the term Hopf gebra as synonym for bi-associative, not necessarily bi-connected antipodeal convolution. This is a bi-algebra as we will show below.

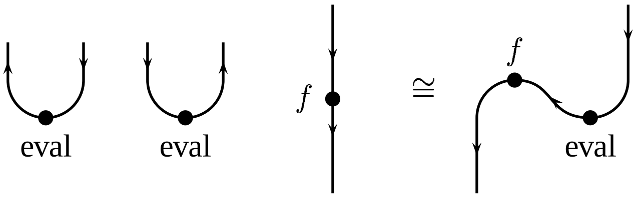
Note that at every point the arrows describe *yes* or *no* questions which renders the structures in a single line as being disjoint. Hence a Hopf gebra is *not* a unital bigebra, etc. This is a major difference to the usual treatment of Hopf al-gebras, where a Hopf algebra is in the same time a bialgebra, and the notions there are inclusive and not exclusive.

5.1 Cup and cap tangles

5.1.1 Evaluation and co-evaluation

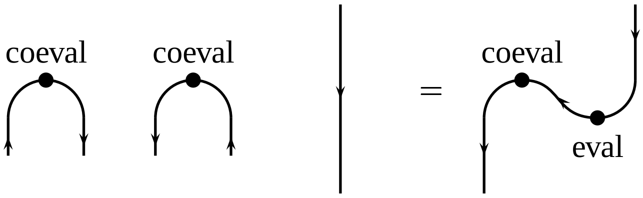
While in the alphabet for knots and links, Eqn. 3-53, cup and cap tangles already occurred, the convolution alphabet, Eqn. 3-57, does not contain such tangles.

However, dealing with endomorphisms, we have implicitly assumed an action of a dual space since $f \in \mathrm{End}\, V \cong \mathrm{Hom}(V, V) \cong V \otimes V^*$. Then, the action of V^* on V is described by a cup tangle, the *evaluation map* of type $2 \to 0$ denoted as $eval\ :\ V^* \otimes V \to \Bbbk$. V or elements x of V are represented by downward pointing arrows, while V^* or elements ω from V^* are drawn as upward pointing arrows, i.e. we use oriented lines which already occurred in the Kuperberg graphical method.



left action by evaluation (5-2)

Co-evaluation is displayed by a cap tangle. This can be seen considering the identity map in the above given description:



(5-3)

Cup and cap tangles constitute a so called *closed structure* [78, 89].

Having introduced an irrelevant basis $\{e_i\}$ in V and a canonical dual basis $\{\epsilon^j\}$ in V^*, i.e. $\epsilon^i(e_j) = \delta^i_j$, one can easily compute the action of $\omega \in V^*$ on $v \in V$ using $(\omega_i \epsilon^i)(v^j e_j) = \omega_i v^j \epsilon^i(e_j) = \omega_i v^j \delta^i_j = \omega_i v^i$. However, this relation does *not* introduce a duality operation $*(e_i) = \epsilon^i$. This isomorphism, simply keeping the coefficients, is called Euclidean dual isomorphism [121, 122]. If we introduce Graßmann exterior algebras one has to define the action of a multi-co-vector $\in \bigvee V^*$ on multivectors $\in \bigwedge V$, where we used a *vee* $\vee$ to denote the exterior product of co-vectors. Following standard conventions [18, 130, 43, 119] one introduces the following *pairing* on homogenous elements (extensors) and extends it by bilinearity

$$
\begin{aligned}
&\langle\,|\,\rangle \;:\; \bigvee V^* \times \bigwedge V \to \Bbbk \\
&\langle V^* \mid V\rangle \;\cong \mathrm{eval} \\
&\langle \omega_1 \vee \ldots \vee \omega_n \mid x_1 \wedge \ldots \wedge x_m\rangle = \begin{cases} \pm\det(\langle \omega_i \mid x_j\rangle) & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\tag{5-4}
$$

The $\pm$ sign has to be arranged due to the involved permutations. We use sometime a slightly different setting, where the indices in the first argument are in reversed order and no sign occurs in front of det. Using this construction the space underlying the Graßmann algebra can be turned into a Hilbert space [41]. In fact this is nothing but the Laplace expansion. In Hopf algebraic terms, see [119], the pairing can be expanded as

$$
\begin{aligned}
\langle \omega' \vee \omega'' \mid x \wedge y\rangle &= \langle \omega' \otimes \omega'' \mid \Delta_\vee(x \wedge y)\rangle \\
&= \langle \omega' \mid (x \wedge y)_{(1)}\rangle\langle \omega'' \mid (x \wedge y)_{(2)}\rangle \\
&= \langle \omega' \mid x_{(1)}\rangle\langle \omega'' \mid y_{(2)}\rangle + (-)^{\partial x \partial y}\langle \omega' \mid y_{(1)}\rangle\langle \omega'' \mid x_{(2)}\rangle
\end{aligned}
\tag{5-5}
$$

Note that since wedge and vee are independent, $(\Delta_\vee, \wedge)$ is also a pair of an independent Graßmann co-product and product. If $\omega', \omega'' \in V^*$ and $x, y \in V$, this is the particular case of a 2×2-determinant.

5.1.2 Scalar and co-scalar products

To be able to introduce Clifford algebras and Clifford co-gebras, we need to introduce scalar and co-scalar products. Let $B \in V^* \otimes V^*$ be a scalar product and $D \in V \otimes V$ be a co-scalar product:

$$V \underset{D}{\overset{B}{\rightleftarrows}} V^* \tag{5-6}$$

Equivalently using the action by evaluation this can be denoted as

$$V \otimes V \xrightarrow{\;B\;} \Bbbk \xleftarrow{\;D\;} V^* \otimes V^* \tag{5-7}$$

These actions are depicted also by cup and cap tangles, but with two ingoing and two outgoing lines. The tangles are also decorated by the map in use

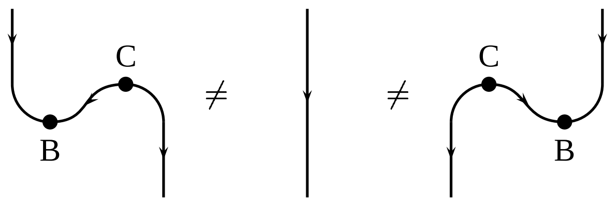
(5-8)

Using categorial duality one introduces the corresponding cap tangles

$$V \otimes V \xleftarrow{\;C\;} \Bbbk \xrightarrow{\;E\;} V^* \otimes V^* \tag{5-9}$$

(5-10)

Observe, that $C^{-1} \neq D$ and $E^{-1} \neq B$ in general. That means, that also the Reidemeister moves are not in general valid and the present tangles are not 'knottish', e.g.:



(5-11)

The condition for Reidemeister moves to hold is hence $C \circ B = Id_V$ and $D \circ E = \mathrm{Id}_{V^*}$. In the case of Clifford products, we will learn, that this condition prevents the existence of an antipode.

5.1.3 Induced graded scalar and co-scalar products

Till now, we have not made any assumptions about the scalar and co-scalar products. However, since we will deal mainly with Clifford algebras, our particular scalar and co-scalar products will have a quite special structure.

It is convenient to introduce the following rules for a *pairing* of n-co-tensors on n-tensors

$$\begin{aligned} &\langle V^* \mid V \rangle \to \Bbbk \\ &\langle V^* \otimes \ldots \otimes V^* \mid V \otimes \ldots \otimes V \rangle \to \Bbbk \end{aligned} \tag{5-12}$$

and to agree that the pairing of m-co-tensors on n-tensors is zero for $m \neq n$. This notion suggests that we index n-co-tensors in a reverse way as n-tensors, i.e.

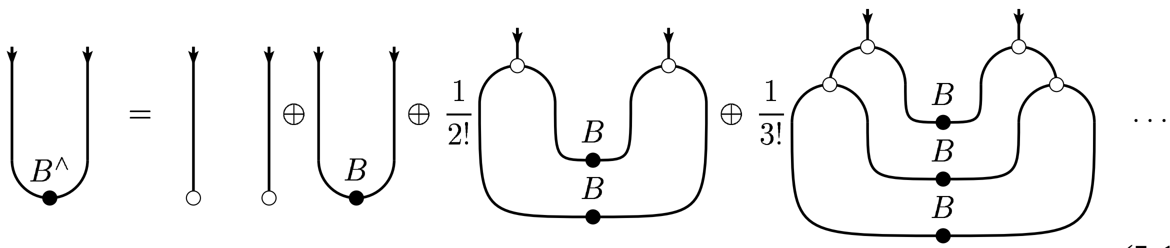
$$\langle \omega_n \otimes \ldots \otimes \omega_1 \mid x_1 \otimes \ldots \otimes x_n \rangle = \langle \omega_1 \mid x_1 \rangle \ldots \langle \omega_n \mid x_n \rangle \,. \tag{5-13}$$

This setting reflects the practice in the theory of knots and links, where the tangles are closed by cup and cap tangles of adjacent open ends of a braid to form a knot or link. The above pairing between antisymmetric exterior products is the Graßmann Hopf version of this definition.

Now let us turn to the case of Graßmann Hopf algebras. Let V be a (finitely generated) $\Bbbk$-module and $\bigwedge V$ the Graßmann algebra built over this space. Furthermore let V^* be the dual space and $\bigvee V^*$ the Graßmann algebra over that space. Usually one introduces there a bilinear form, i.e. a scalar product, $B\,:\; V \otimes V \to \Bbbk$, or a bilinear form, i.e. a co-scalar product, $C\,:\; V^* \otimes V^* \to \Bbbk$. The question arrises, in which way the bilinear forms are lifted to the whole space $\bigwedge V$ or $\bigvee V^*$. Let us denote this lifted scalar and co-scalar products by $B^\wedge$ and $C^\vee$

Remember that $B \in V^* \otimes V^*$, and that $B^\wedge$ will live in $\bigvee V^* \otimes \bigvee V^*$. We require that this extension is a graded morphism $B^\wedge \in \mathrm{Hom}(\bigwedge V \otimes \bigwedge V, \Bbbk)$. The required extension can be given, see Oziewicz [102, 99] and [128, 60, 17], as:

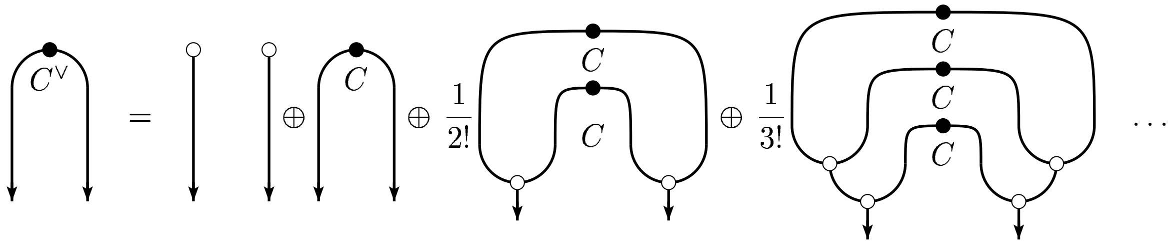
$$B^\wedge = \exp(B) = \epsilon \otimes \epsilon + B_{ij}\epsilon^i \otimes \epsilon^j + B_{[i_1 i_2],[j_1 j_2]}\epsilon^{i_1} \wedge \epsilon^{i_2} \otimes \epsilon^{j_1} \wedge \epsilon^{j_2} + \ldots$$



(5-14)

And the same holds true for co-scalar products:

$$C^\vee = \exp(C) = \eta \otimes \eta + C^{ij} e_i \otimes e_j + C^{[i_1 i_2],[j_1 j_2]} e_{i_1} \wedge e_{i_2} \otimes e_{j_1} \wedge e_{j_2} + \ldots$$



(5-15)

Note that in the r.h.s inner lines B and C act on grade 1 spaces only. Therefore $B^\wedge$ and $C^\vee$ are graded extensions of B and C.

The combinatorial factors $1/n!$ are not apparent if one has already taken account for the antisymmetry in terms like $B_{[i_1,i_2],[j_1j_2]} = B_{i_1j_1}B_{i_2j_2} - B_{i_1j_2}B_{i_2j_1}$. We introduce for the scalar and co-scalar product if extended to the whole space $\bigwedge V$ or $\bigvee V^*$ also the Sweedler notation

$$
\begin{aligned}
B^\wedge &= B^\wedge_{(1)} \otimes B^\wedge_{(2)} \\
C^\vee &= C^\vee_{(1)} \otimes C^\vee_{(2)}\,.
\end{aligned}
\tag{5-16}
$$

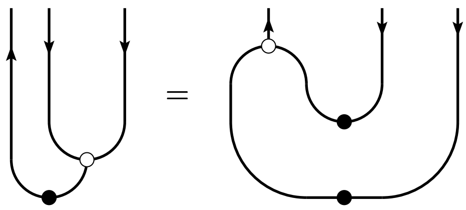
We note furthermore that for the Clifford co-product of Id, based on the co-scalar product C defined below, one finds

$$
\Delta_C(\mathrm{Id}) = C^\vee = C^\vee_{(1)} \otimes C^\vee_{(2)}. \tag{5-17}
$$

5.2 Product co-product duality

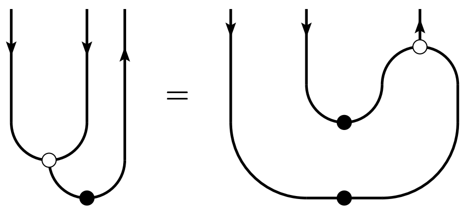
5.2.1 By evaluation

Having the evaluation established, we can explain the important concept of product co-product duality. Observe, that a co-vector might act on a *product* of 2 vectors $\omega(ab)$ and one can ask if the co-vector can be 'distributed' on a and b. Using tangles we obtain



(5-18)

That is, one obtains $\omega(ab) = (\mathrm{eval} \otimes \mathrm{eval})\Delta(\omega)(a \otimes b) = \omega_{(1)}(b)\omega_{(2)}(a)$, where the co-product $\Delta = m^*$ is the dualized product. Indeed, this can be done the other way around also



(5-19)

which shows how a product of co-vectors $\omega'\omega''$ can be distributed over a vector v as $(\omega'\omega'')(v) = (\mathrm{eval} \otimes \mathrm{eval})(\omega' \otimes \omega'')(v_{(1)} \otimes v_{(2)}) = \omega''(v_{(1)})\omega'(v_{(2)})$.

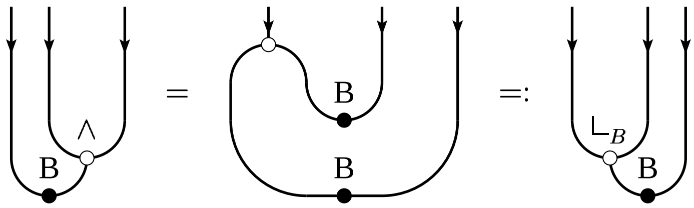
In fact, the statement that an algebra over A is dualized by categorial duality into a co-algebra over A and vice versa is an equivalent assertion. The importance of this construction cannot be overemphasized, since the whole theory of determinants, permanents and their generalizations to supersymmetric spaces can be developed from this setting, [66]. Furthermore, as we will demonstrate below this type of duality also *yields* commutation relations.

5.2.2 By scalar products

Using the evaluation, we compose co-vector spaces and vector spaces, which made it necessary to put arrows on the tangles. Since we have introduced cup and cap tangles for scalar and co-scalar products, one can proceed to introduce co-products from products, which are derived from these tangles and where the entries are of the same type. However, this is no longer a duality in the above defined sense since it involves explicitely a scalar or co-scalar product. We will see, that an entirely new type of product will occur, the *contraction*. Due to our construction of the scalar product $B^\wedge$ as a graded morphism, $B^\wedge : \bigwedge V \to \bigvee V^*$, we have the important relation

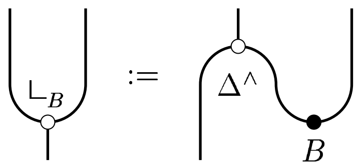
$$B^\wedge(a \wedge b) = B^\wedge(a) \vee B^\wedge(b). \tag{5-20}$$

This was called outermorphism by Hestenes and Sobczyk [69]. The tangle equation is once more



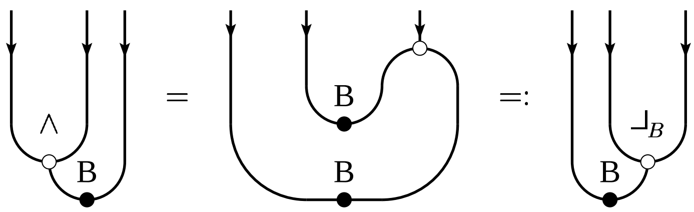
(5-21)

where we have defined the new product $\llcorner_B$, i.e. right contraction w.r.t. B. The defining tangle of the *right contraction* is thus:



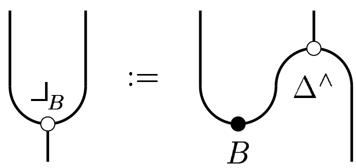
(5-22)

Of course, we can define in an analogous way the left contraction



(5-23)

which leads to the defining tangle for the *left contraction*:



(5-24)

Note that these relations for the left and right contraction are valid on the whole graded A-module, i.e. for *any grade* and inhomogeneous element. It is to the best knowledge of the author the first time that such a formula is explicitely given. The same feature will be observed with the Rota-Stein cliffordization. We will identify below in the physics section contractions with respect to a 1-vector as *annihilation* operators and the wedge product with a 1-vector as *creation* operators

$$v \rfloor \;\cong\; a_v \qquad\qquad v\wedge \;\cong\; a_v^\dagger \tag{5-25}$$
$$e_i \rfloor \;\cong\; a_i \qquad\qquad e_i\wedge \;\cong\; a_i^\dagger \quad . \tag{5-26}$$

The above given generally valid relation will have an impact on calculations in quantum field theory.

In fact, $B^\wedge$ and $C^\vee$ can be seen to be dual isomorphisms of a new kind. If one has a pairing $\langle . \mid . \rangle_{\text{eval}}$, it is clear that vectors are moved to co-vectors by keeping the coefficients and altering the basis:

$$\langle \omega \mid v \rangle_{\text{eval}} = \omega(v) = \langle \epsilon^0 \mid \omega^\star \rfloor_\delta \, v \rangle_{\text{eval}} = \omega^{\star i} v_i \langle \epsilon^0 \mid e_0 \rangle_{\text{eval}}, \tag{5-27}$$

where $\omega = \omega_i \epsilon^i$ and $\omega^\star = \omega^{\star i} e_i = \omega_j \delta^{ji} e_i$. Since there is no canonical dual isomorphism, it is quite artificial to use δ. If a scalar product and a co-scalar product are given, it is natural to use these maps B and C to move vectors to co-vectors and vice versa in a pairing. We denote $B(u)$ the co-vector image of u under the map B and $C(\omega)$ the vector image of ω under the map C.

$$\langle \omega \mid u \wedge v \rangle_{\text{eval}} = \langle \omega \llcorner_\delta \, B(u) \mid v \rangle_{\text{eval}} = \langle \omega \llcorner_B \, u^\star \mid v \rangle_{\text{B,C}} \tag{5-28}$$

and

$$\langle \omega \vee \rho \mid v \rangle_{\text{eval}} = \langle \omega \mid C(\rho) \rfloor_\delta \, v \rangle_{\text{eval}} = \langle \omega \mid \rho^\star \rfloor_C \, v \rangle_{\text{B,C}}. \tag{5-29}$$

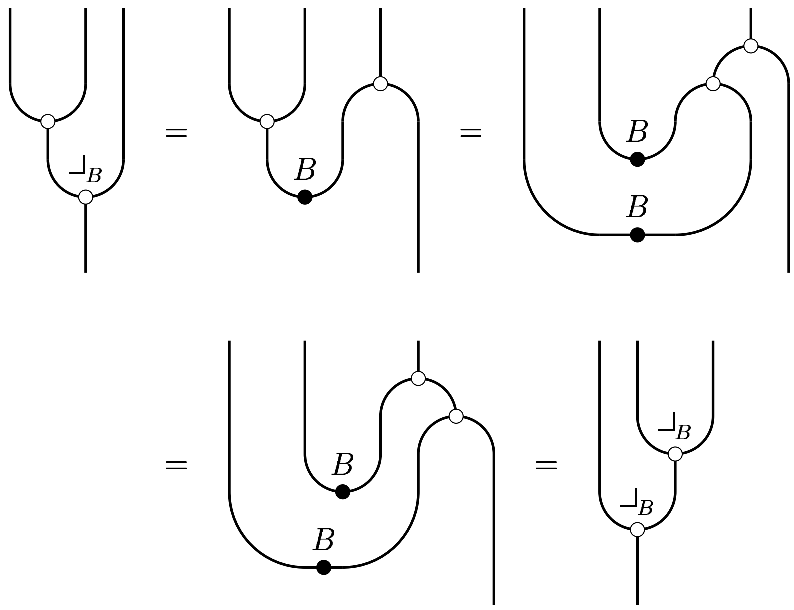
This setting yields exactly the graded extension $B^\wedge$ and $C^\vee$ as introduced above. In physics, this will ensure that the adjoint of a creation operator will be an annihilation operator and not a polynomial of annihilation operators. Such a non-graded extension of B will lead to *polarization effects*. Note that we have not assumed that $B \circ C = \mathrm{Id}_V$ and hence the dualized dual is not in general identical with the original element.

Let us explore the calculation rules of the tangle Eqns. 5-22 and 5-24. If we compute the left contraction on two 1-vectors a and b, we get:

$$\begin{aligned}
\rfloor_B(a \otimes b) &= (B \otimes \mathrm{Id})\left((\mathrm{Id} \otimes \Delta)(a \otimes b)\right) \\
&= (B \otimes \mathrm{Id})(a \otimes b \otimes Id + a \otimes \mathrm{Id} \otimes b) \\
&= B(a,b)\mathrm{Id}\,.
\end{aligned} \tag{5-30}$$

That is on 1-vectors the contraction product simply evaluates to $\eta \circ B$. That is the first law of Chevalley deformation. Two further relations are required which describe how the contraction

’distributes’ on the primary product. We compute firstly the *left straightening law*:



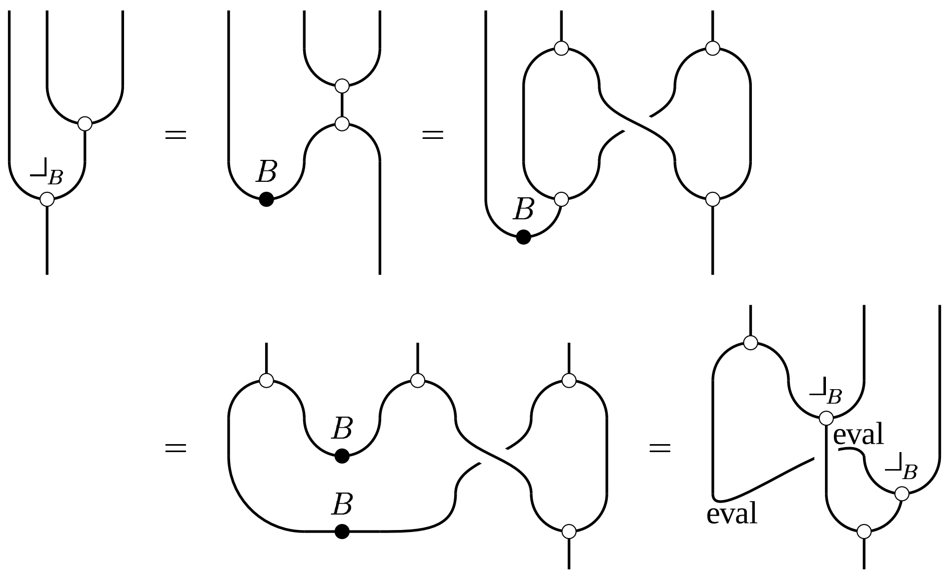
(5-31)

That is, from the co-associativity of Δ and product co-product duality w.r.t. B we have *derived* the rule

$$(u \wedge v) \lrcorner_B w = u \lrcorner_B (v \lrcorner_B w), \tag{5-32}$$

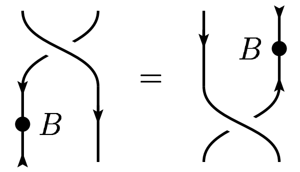
where u, v, w are arbitrary elements from $\bigwedge V$. This is the third law of Chevalley deformation.

To compute the *right straightening law* we have to compute



(5-33)

where we had to assume the identity



(5-34)

which is a requirement on the crossing. If this identity does not hold, we cannot move the $\rfloor_B$ product 'under' the crossing but get not a contraction but a different product. The above used identity reads in algebraic terms

$$B^s_c \tau^{ab}_{sd} = \tau^{ab}_{cs} B^s_d. \tag{5-35}$$

The tangle equation 5-33 reads algebraically for arbitrary multivectors u, v, w

$$w \rfloor_B (u \wedge v) = (-1)^{(|w_{(1)}||w_{(2)} \rfloor_B u|)} \left(w_{(2)} \rfloor_B u\right) \wedge \left(w_{(1)} \rfloor_B v\right) \tag{5-36}$$

If we assume the leftmost input w to be a 1-vector a, we arrive with Graßmann products and the Graßmann graded switch as crossing, where Eqn. 5-34 holds, at the following formula

$$a \rfloor_B (u \wedge v) = (a \rfloor_B u) \wedge v + \hat{u} \wedge (a \rfloor_B v), \tag{5-37}$$

which is the second law of Chevalley deformation. However, our Hopf algebraic result in Eqn. 5-33 is valid for the input of any element of any grade, even inhomogeneous. The crossing has been replaced by $\hat{u}$, the grade involution. This is possible only if the first factor is a 1-vector and shows that Chevalley deformation is restricted by the fact that it does not properly deal with the crossing:

$$\begin{aligned} \hat{\tau}(a \otimes u) &= (-1)^{\partial a \partial u}(u \otimes a) = ((-1)^{\partial u} u) \otimes a \\ &= \hat{u} \otimes a. \end{aligned} \tag{5-38}$$

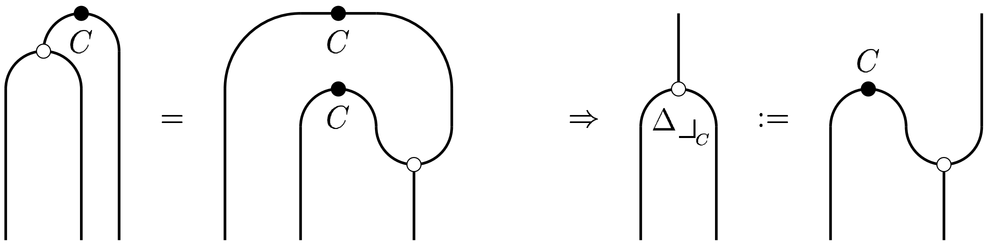
Summarizing the formulas which we have just derived, we end up with the rules of Chevalley deformation of a Graßmann algebra, i.e. a Clifford map which is given on 1-vectors $x \in V$ as

$$x \to \gamma_x := x \rfloor_B + x \wedge . \tag{5-39}$$

The operator $\gamma \; : \; V \otimes \bigwedge V \to \bigwedge V$ can be lifted to an action $\gamma \; : \; \bigwedge V \otimes \bigwedge V \to \bigwedge V$ by *recursive application* and linearity. However, the Hopf algebraic counterparts are valid on the whole space and do not suffer any restriction on their input. We can summarize the formulas which we have derived from Hopf gebraic considerations and compare them to the literature, e.g [31, 18, 27, 40, 87]. Let $a, b \in V$, $u, v, w \in \bigwedge V$ it holds:

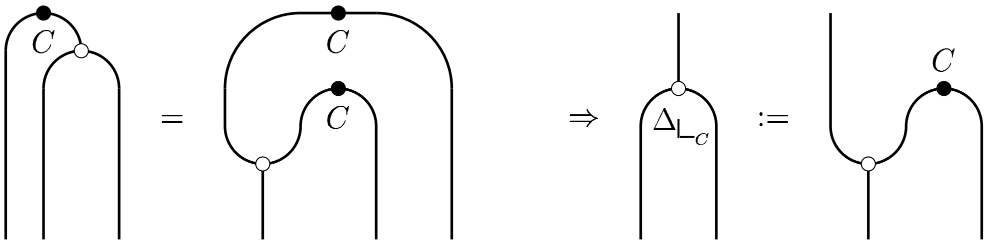
$$\begin{aligned} a \rfloor_B b &= B(a, b) \\ a \rfloor_B (u \wedge v) &= (a \rfloor_B u) \wedge v + \hat{u} \wedge (a \rfloor_B v) \\ (u \wedge v) \rfloor_B w &= u \rfloor_B (v \rfloor_B w) \end{aligned} \tag{5-40}$$

As a matter of fact, we could now develop the *co-contraction*, using the co-scalar product C and derive analogous relations as Eqns. 5-40.



(5-41)

and



(5-42)

In Sweedler notation these formulas are displayed as – remember that $C^\wedge = \Delta_C(\mathrm{Id})$:

$$
\begin{aligned}
\Delta_{\lrcorner_C}(x) &= C^\wedge_{(1)} \otimes (C^\wedge_{(2)} \wedge x) \\
\Delta_{\llcorner_C}(x) &= (x \wedge C^\wedge_{(1)}) \otimes C^\wedge_{(2)}
\end{aligned}
\tag{5-43}
$$

It would be now possible to derive a co-Chevalley deformation based on a co-Clifford map.

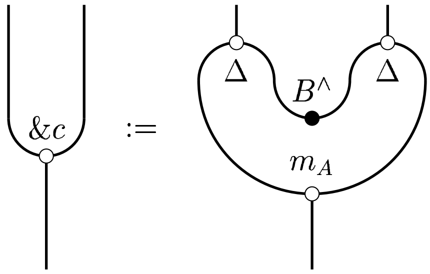
5.3 Cliffordization of Rota and Stein

Cliffordization is a quite remarkable process. A product or co-product is *deformed* by cliffordization to yield a new *quantized* product, see [101]. Deformation and quantization are therefore intimately related. In fact it turns out that imposing non-trivial commutation relations is equivalent to the choice of a bilinear form which gives rise to the deformation. It is remarkable to note that the process of cliffordization is quite ubiquitous in mathematics and not restricted to quantum physics, [119, 118]. Since we will use cliffordization mainly for Graßmann exterior algebras it should be emphasized that this method works for symmetric algebras and even more general algebras also.

5.3.1 Cliffordization of products

Let $m_A\ :\ A \otimes A \to A$ be the product of a Hopf algebra $H(A, m_A, \eta, \Delta, \epsilon; S)$, or a convolution $\mathrm{Conv}(A, A)$. As a prototype, the reader may think of a Graßmann wedge product. Now let a scalar product $B^\wedge$ be given on A by exponentiation of B, which is represented by a cup tangle.

Definiton 5.1 (Cliffordization). *A* Clifford product *(or circle product) $\&c$ on A is defined via the tangle*



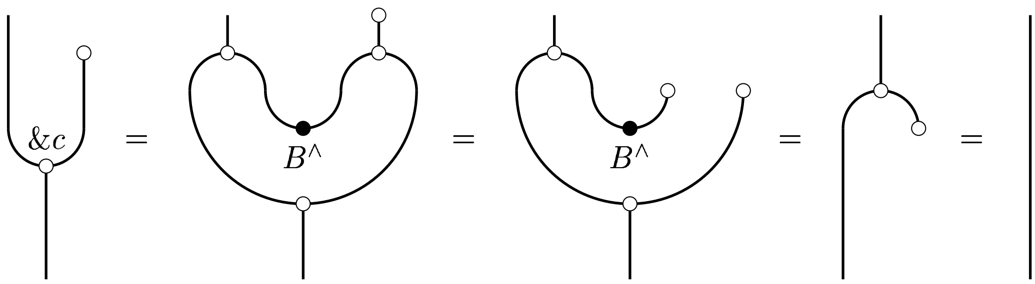
(5-44)

where $B^\wedge$ is the bilinear form obtained from $B : V \otimes V \to \Bbbk$ by exponentiation.

This tangle can be easily remembered as 'sausage tangle', the term was coined by Oziewicz [103]. Of course, since $\&c$ is a $2 \to 1$ map, it is a product. The new product is totally defined by the structure tensors Δ, m_A and the scalar product B of the primarily given Hopf algebra or convolution. Cliffordization is a quite general process, see [119, 118] and it is by no means restricted to Graßmann Hopf algebras. It needs, in principle, only a convolution and a scalar product or even less restrictive two product maps.

Theorem 5.2. *The unit η of m_A, if it exists, remains to be the unit of the cliffordized product if (i) the unit η is a co-algebra homomorphism of the primary co-product, and, (ii) the unit η and counit ϵ are related via $B(A \otimes \eta) = \epsilon = B(\eta \otimes A)$.*

Proof: The proof is given for the unit multiplied from the right, the left multiplication case can be shown analogously.

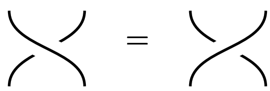


(5-45)

In fact this means that we are dealing with an augmented algebra and an augmented co-algebra as primary structure.

Theorem 5.3. *The cliffordized product $\&c$ of a bi-associative Hopf gebra or a bi-associative bigebra is associative under the condition that the crossing fulfils the following symmetry requirement*

$$\tau^{ab}_{cd} = \tau^{bd}_{ac}$$



(5-46)

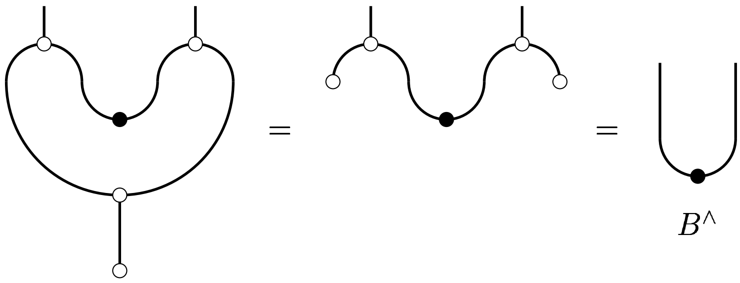
Proof: Looking at the tangles and using the fact that m_A is an co-algebra homomorphism and Δ is an algebra homomorphism and product co-product duality between m_A and Δ yields the result.

Rota and Stein showed, that the new structure $H(A, \&c, \eta, \Delta, \epsilon; S^c)$ is a Hopf algebra. A point which was criticized at this level is that the product is deformed by the sausage tangle, but the co-product remains to be the same. This is quite unnatural. Moreover, since these products are no longer related by product co-product duality, a further cliffordization may lead to non-associative products. This motivates the following definition:

Definiton 5.4 (local and non-local products). *A product which possesses an augmentation ϵ such that ϵ is an algebra homomorphism is called* local. *If the augmentation is not an algebra homomorphism the product is called* non-local. *The same notion is introduced by duality for co-products.*

Theorem 5.5. *Products which arise from the process of cliffordization are in general non-local.*

Proof:



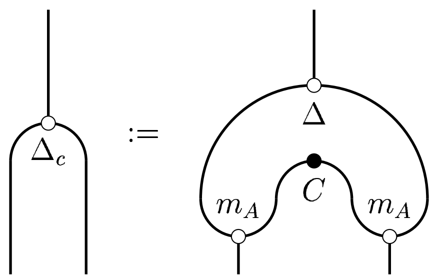
(5-47)

where the last equality holds if the augmentation ϵ is the counit of the primary co-product Δ. Of course, locality is preserved in the trivial case that $B^\wedge = \epsilon \otimes \epsilon$ holds true. But this does not lead to a new product and is not a proper cliffordization.

5.3.2 Cliffordization of co-products

Having studied the cliffordization of products in some detail, we can shortly display the *co-cliffordization*. Let $\Delta : A \to A \otimes A$ be a co-product of a Hopf algebra $H(A, m_A, \eta, \Delta, \epsilon; S)$ of a convolution $\mathrm{Conv}(A, A)$. Let a co-scalar product $C^\vee$ be given, then we define

Definiton 5.6 (Co-cliffordization). *A* co-Clifford product Δ_c *on A is defined via the following tangle. We employ the extension of the co-scalar product $C^\vee$, i.e. a cap tangle:*



(5-48)

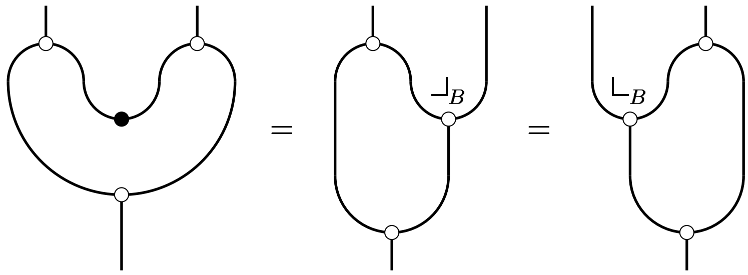
It would be worth to explore the structure $H(A, m_A, \eta, \Delta_c, \epsilon; S^c)$ in the same way as Rota and Stein did for the cliffordization. However, we will concentrate on the case where both products are deformed obtaining this asymmetric deformation as a special case.

In an analogous sense, the definition 5.4 and the assertions 5.2, 5.3 and 5.5 can be established for Clifford co-products too.

5.3.3 Clifford maps for any grade

A major drawback of Chevalley deformation of Graßmann algebras is that the Clifford map $\gamma : V \otimes \bigwedge V \to \bigwedge V, x : x \to \gamma_x$ is defined on 1-vectors only.

However, since we found tangles for contractions of any grade, we can now define a Clifford map for any grade using Hopf algebraic techniques. Observe that the Rota and Stein 'sausage' tangle of cliffordization can be rewritten as



(5-49)

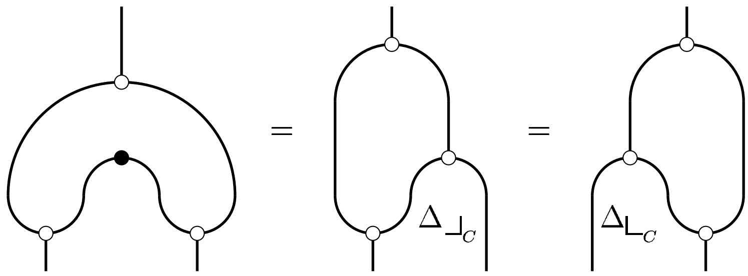
On 1-vectors this results in

$$
\begin{aligned}
a \,\&c\, b &= a_{(1)} \wedge (a_{(2)} \lrcorner_B\, b) \\
&= a \wedge (\mathrm{Id} \lrcorner_B\, b) + \mathrm{Id} \wedge (a \lrcorner_B\, b) \\
&= a \wedge b + a \lrcorner_B\, b = \gamma_a\, b \qquad (5\text{-}50) \\
a \,\&c\, b &= (a \llcorner_B\, b_{(1)}) \wedge b_{(2)} \\
&= (a \llcorner_B \mathrm{Id}) \wedge b + (a \llcorner b) \wedge \mathrm{Id} \\
&= a \wedge b + b \llcorner_B\, a = \gamma_a\, b. \qquad (5\text{-}51)
\end{aligned}
$$

Now, the above formula holds also in higher grades, and even when elements are inhomogeneous. We compute an example where $a, b, x, y \in V$ and we Clifford multiply two step 2 extensors

$$
\begin{aligned}
(a \wedge b) \,\&c\, (x \wedge y) &= (a \wedge b) \wedge (x \wedge y) + a \wedge (b \lrcorner_B ((x \wedge y)) \\
&\quad - b \wedge (a \lrcorner_B ((x \wedge y)) + \mathrm{Id} \wedge ((a \wedge b) \lrcorner_B (x \wedge y)) \\
&= \quad a \wedge (b \wedge (x \wedge y) + b \lrcorner_B (x \wedge y)) \\
&\quad + a \lrcorner_B (b \wedge (x \wedge y) + b \lrcorner_B (x \wedge y)) - (a \lrcorner_B b)(x \wedge y) \\
&= \gamma_a(\gamma_b(x \wedge y)) - (a \lrcorner_B b)(x \wedge y) \\
&= (\gamma_a \wedge \gamma_b)(x \wedge y)\,. \qquad (5\text{-}52)
\end{aligned}
$$

The case of co-cliffordization is handled along the same lines. We get



(5-53)

and see that one can introduce a co-Clifford map. One finds obviously for an element a of *any* grade, even inhomogeneous,

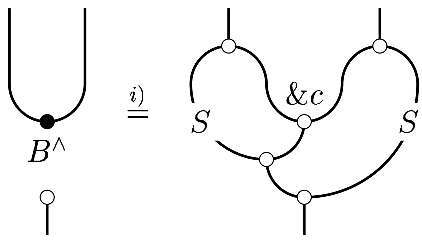
$$\begin{aligned}\Delta_C(a) &= (\wedge \otimes \Delta_{\lrcorner_C})(a_{(1)} \otimes a_{(2)}) \\ &= (a_{(1)} \wedge C_{(1)}) \otimes (C_{(2)} \wedge a_{(2)})\,. \end{aligned} \tag{5-54}$$

5.3.4 Inversion formulas

A quite interesting point remains to be examined. Is it possible to invert the cliffordization process. That is, given a deformed or cliffordized product $\&c$ or co-cliffordized co-product Δ_c, one can obtain back the undeformed product m_A or co-product Δ. This is done by the Rota and Stein inversion formulas [119], section 4, p.13059. In our notation using Sweedlers convention about co-products, we find for Graßmann-Clifford products:

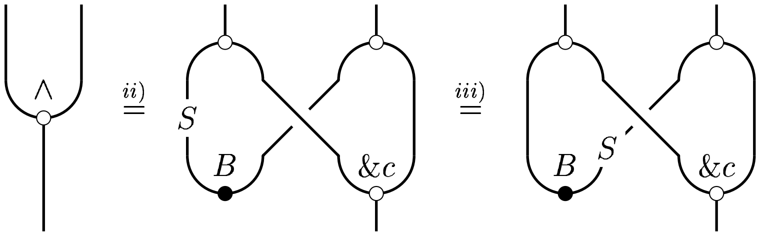
$$\begin{aligned}
&i) && B(u,v) = \sum_{(u)(v)} S(u_{(1)}) \wedge (u_{(2)} \,\&c\, v_{(1)}) \wedge S(v_{(2)}) \\
&ii) && u \wedge v = \sum_{(u)(v)} \pm B(S(u_{(1)}), v_{(1)})(u_{(2)} \,\&c\, v_{(2)}) \\
&iii) && u \wedge v = \sum_{(u)(v)} \pm B(u_{(1)}, S(v_{(1)}))(u_{(2)} \,\&c\, v_{(2)})
\end{aligned} \tag{5-55}$$

where S is the antipode of the undeformed Hopf algebra. The tangles of this relations read:



(5-56)

and



(5-57)

It will be of great value for the later discussed applications to provide a few examples of these formulas. Let $x, y, z \in V$, one finds:

$$
\begin{aligned}
B(x,y) &= x \,\&c\, y - x \wedge y \\
x \wedge y \wedge z &= x \,\&c\, (y \wedge z) - B(x,y)z + B(x,z)y \\
&= x \,\&c\, y \,\&c\, z - B(x,y)z + B(x,z)y - B(y,z)x.
\end{aligned}
\tag{5-58}
$$

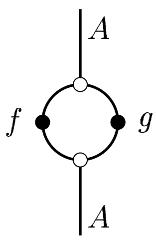
These are the basic formulas which have been employed in [48, 50, 56] to perform vertex normal-ordering, as it will be discussed below.

The most remarkable fact is, that for the inversion formulas to hold one needs to have an antipode S. It is also the antipode which is needed in the Connes-Kreimer renormalization method, i.e the antipode is hidden in the BPHZ formalism of perturbative renormalization. This gives strong evidence that quantum field theory should be formulated with Hopf gebras.

5.4 Convolution algebra

We have already defined the convolution using the structure tensors Δ and m which are assumed to be associative here. We restrict our discussion to the endomorphic case.

$$
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\star"'] & A \otimes A \arrow[d, "f \otimes g"] \\
A & A \otimes A \arrow[l, "m"']
\end{tikzcd}
$$



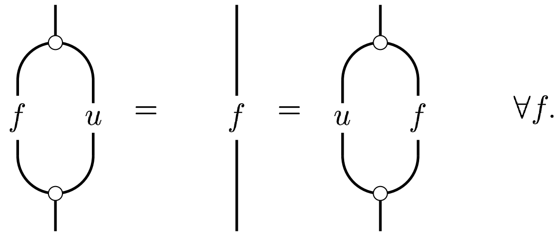
(5-59)

This defines the convolution algebra $\mathrm{Conv}(A, m, \Delta)$ on the endomorphisms $f : A \to A$. The convolution product is denoted by $\star : \mathrm{End}\, A \otimes \mathrm{End}\, A \to \mathrm{End}\, A \cong A \otimes A^*$.

A convolution unit u is defined as usual,

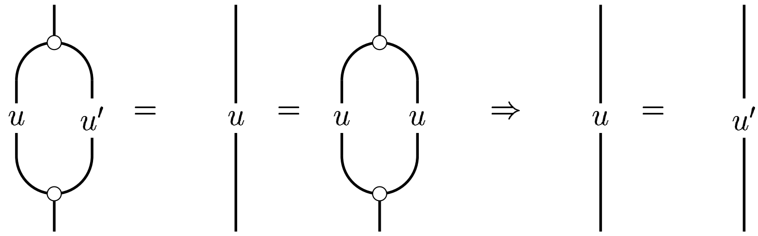
$$
f \star u \;=\; f \;=\; u \star f \tag{5-60}
$$

or in terms of tangles



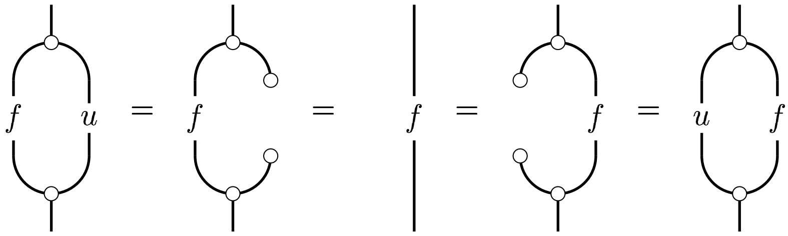
(5-61)

The unit, if it exists, is unique, since we find for two units u and u',



(5-62)

If the product m possesses a unit η and the co-product possesses a counit ϵ, then the convolution unit u is given by $u = \eta \circ \epsilon$, since we find



(5-63)

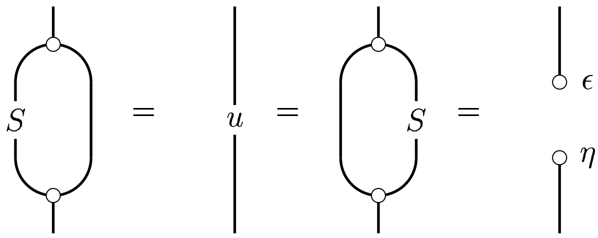
where we used the unit and counit properties displayed in Eqns. 4-3 and 4-22.

Since we are interested in Graßmann and Clifford Hopf gebras, we are dealing with unital associative algebras and co-gebras. From product co-product duality it follows, that unital algebras are related to counital co-gebras.

Theorem 5.7. *If m is a Graßmann (Clifford) product and Δ is a Graßmann (Clifford) co-product, then $\star$ is a unital convolution with unit $u = \eta \circ \epsilon$.*

Note that we use product co-product duality to make statements about the character of the mutual structure, we do *not* however assert the relation $m^* = \Delta$ or equivalently $\Delta^* = m$.

Having established that Graßmann or Clifford bi-convolutions are unital, we can ask if an antipode exists. Recalling the axioms:



(5-64)

we can prove that the Graßmann bi-convolution is antipodal. Therefore a Graßmann algebra is graded and augmented, by the counit, and that it is connected. Thus the recursive proof for the existence of teh antipode from page 62 applies.

This argument does not apply to Clifford algebras. Clifford algebras are filtered algebras w.r.t. the $\mathbb{Z}_n$-filtration inherited from the Graßmann algebra when viewed as endomorphism (sub)algebra of $\bigwedge V$ or inherited from the tensor algebra via factorization. The Clifford product is $\mathbb{Z}_2$-graded only, and the counit if taken as augmentation, yields a non-connected algebra.

Theorem 5.8 (Oziewicz 1997 [102]). *A Clifford bi-convolution with product m^B based on the scalar product B and a co-product Δ^C based on the co-scalar product C is antipodal if and only if $C^{-1} \neq B$.*

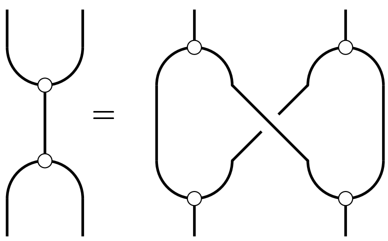
In other words, if one uses the particular co-product which is gained by product co-product duality, the resulting bi-convolution $\mathrm{Conv}(A, m^B, \Delta^{B^{-1}})$ is antipodeless. This seemed to be a great drawback in the study of Clifford Hopf gebras, and led to the study of convolutions with independent product and co-product by Oziewicz and coworkers.

Regarding our analysis and recalling the idea of a Peano space and the Graßmann-Cayley algebra, it is quite natural to introduce *independently* a wedge and a vee exterior product on A and A^*. Using now product co-product duality, one ends up with an independent product and co-product. This fact will be a major part of our analysis of normal-ordering in quantum field theory.

The basic point is that one can perform a cliffordization of a Graßmann algebra w.r.t. a purely antisymmetric bilinear form. The resulting product is again an exterior product, but different from the originally introduced wedge. Such an algebra will be called *quantum Graßmann algebra*, for reasons given below.

5.5 Crossing from the antipode

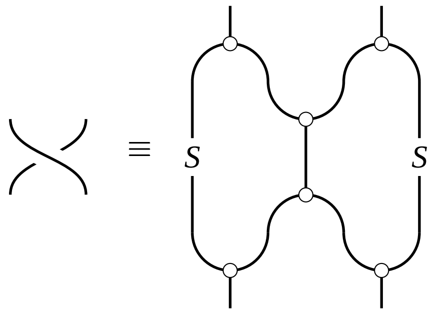
We have up to now identified associative antipodal convolution algebras with Hopf gebras. It remains to show, that the product is a co-gebra homomorphism and that the co-product is an algebra homomorphism. This condition is displayed by the following tangle, compare Eqn. 4-39



(5-65)

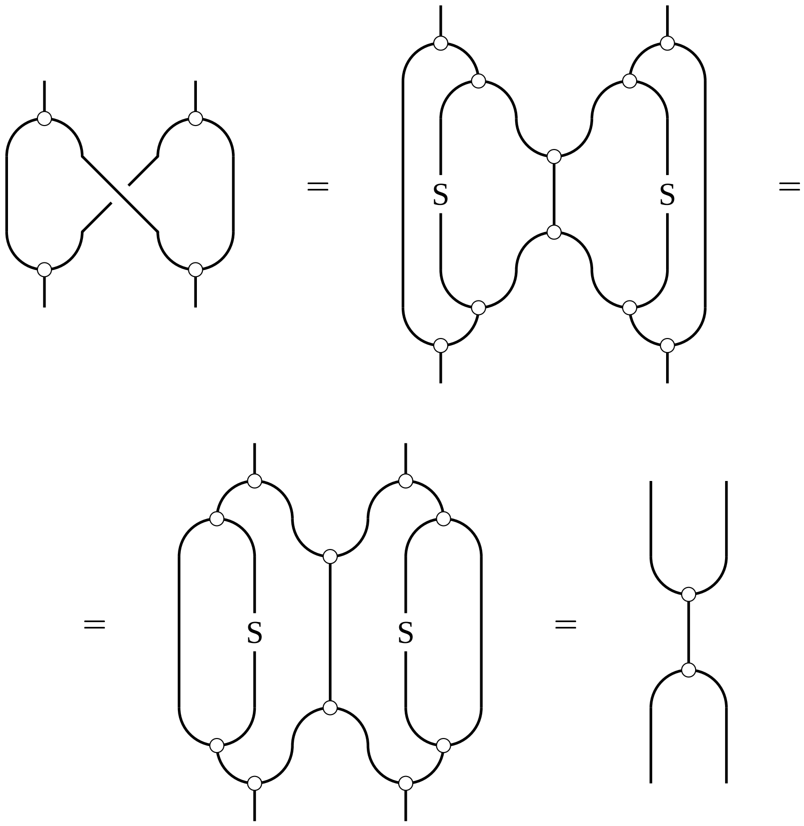
where the crossing occurs in the r.h.s. because of the crossed products involved in that calculation.

Theorem 5.9 (Oziewicz). *The crossing of a bi-associative antipodal bi-convolution is given as*



(5-66)

Proof:



(5-67)

where we have used bi-associativity and the antipode axioms.

In fact, the ’crossing’ is a planar graph, containing no ’over’ or ’under’ information, and has *a priori* nothing to do with knots and links and their projections. This tangle will also not fulfil in general the Reidemeister moves of Eqn. 3-55 or 3-56, even if the considered structure had fulfilled Eqn. 3-54. Hence the name crossing is quite misleading and should probably be replaced with *scattering* or *transmutation*, since it is a generalized switch. However, we will stay with the term crossing since this tangle is employed in building crossed products. It is of utmost importance to classify such crossings. Unfortunately not very much is known till now. For what type of structure tensors m, Δ is the crossing a pre-braid, preserves a grading, filtration etc. ?

Nevertheless, this theorem answers in part our main question about the independence of the structure tensors. We can reformulate it as follows: Given a bi-associative antipodal convolution, then the crossing is a function of the structure tensors m, Δ and S which itself is a function of m, Δ – in physicists notation:

$$\tau = \tau(m, \Delta) \qquad\qquad S = S(m, \Delta). \tag{5-68}$$

Example: We consider a 2-dimensional space V and its dual space V^*. Let $\{e_i\},\ i \in \{1, 2\}$ be an arbitrary basis of V and let $\{\epsilon^j\},\ j \in \{1, 2\}$ be the canonical dual basis of V^* w.r.t. the basis of V defined via $\epsilon^i(e_j) = \delta^i_j$. Introduce a scalar product B and a co-scalar product C as

$$B \cong \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad\qquad C \cong \begin{bmatrix} u & z \\ w & v \end{bmatrix}. \tag{5-69}$$

The Clifford algebra $\mathcal{C}\ell(V, B)$ has a Graßmann basis $GB = \{\mathrm{Id}, e_1, e_2, e_1 \wedge e_2\}$. The co-scalar product induces the following co-product of Id

$$\begin{aligned} \Delta(\mathrm{Id}) &= C^{\wedge}_{(1)} \otimes C^{\wedge}_{(2)} \\ &= \mathrm{Id} \otimes \mathrm{Id} + u\, e_1 \otimes e_1 + z\, e_1 \otimes e_2 + w\, e_2 \otimes e_1 + v\, e_2 \otimes e_2 \\ &+ (zw - uv)\, e_1 \wedge e_2 \otimes e_1 \wedge e_2. \end{aligned} \tag{5-70}$$

Therefrom any co-product can be calculated by Rota-Stein co-cliffordization.

The antipode is a linear operator on $\bigwedge V$ and can be represented in the above defined basis and its dual basis on $\bigvee V^*$. One finds using BIGEBRA [3]

$$\begin{aligned} S = S^a{}_b\, e_a \otimes \epsilon^b \;\cong\; 1/N &\begin{bmatrix} 1 + (c-b)(w-z) & 0 & 0 & -c+b \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ z - w & 0 & 0 & 1 \end{bmatrix} \\ N = (1 - \mathrm{tr}(BC) + \det(BC)\,) & \end{aligned} \tag{5-71}$$

Only if *both* scalar products are symmetric, the antipode is grade preserving, hence a graded morphism. If either one or the other scalar product is symmetric, S has a triangular representation as matrix and preserves the filtration of either $\bigwedge V$ or $\bigvee V^*$. In the case of a quantum Graßmann Hopf algebra, i.e. $B \neq C^{-1}$, $B^T = -B$, $C^T = -C$, one arrives also at an antipode being a non-graded $\Bbbk$-module morphism. This will be of extreme importance in the theory of perturbative renormalization according to Connes-Kreimer, since it flaws the recursive formula for calculating the antipode, see page 62. But this formula, enriched by the renormalization scheme, is equivalent there to the Zimmermann forest formula, [33, 82, 25, 24, 23].

Regarding our example, current computer algebra can derive the crossing, which turns out to be a cumbersome expression, however, it is not able yet to manage to calculate e.g. the minimal polynomial of the derived crossing, or to detect if it is a braid. A special case was, however, discussed in Fauser and Oziewicz [59].

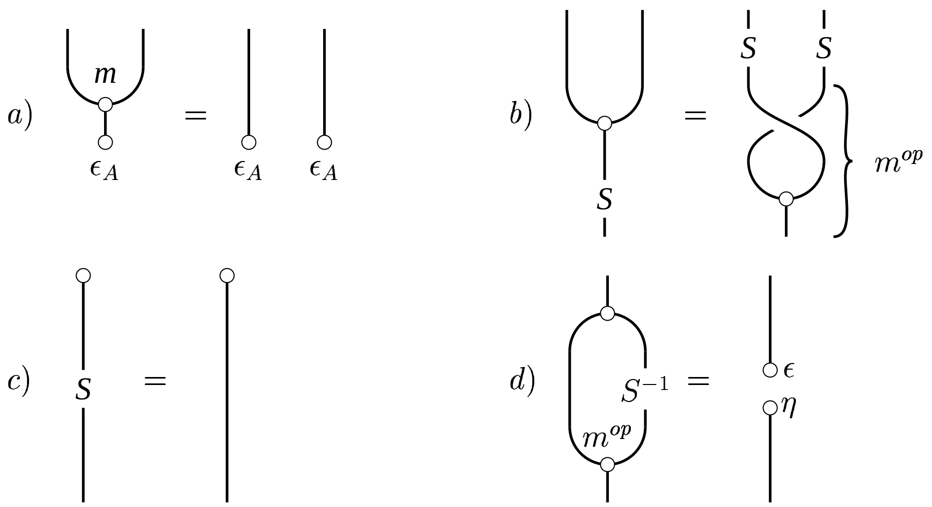
5.6 Local versus non-local products and co-products

5.6.1 Kuperberg Lemma 3.2. revisited

In [84] Kuperberg proved a lemma which establishes some important relations widely used in the theory of 3-manifold invariants, e.g. [76]. Moreover, Kuperberg showed that a certain invariant of Hopf algebras is connected to the Kauffman bracket [75], i.e. the Jones polynomial. This polynomial has however also an impact on quantum field theory as Witten showed [135]. It seems to be generally not well known that Hopf gebras can be defined without being connected. Only recently such structures have been studied by Nill et al. [67, 96, 97], however, we reject the term weak Hopf algebra.

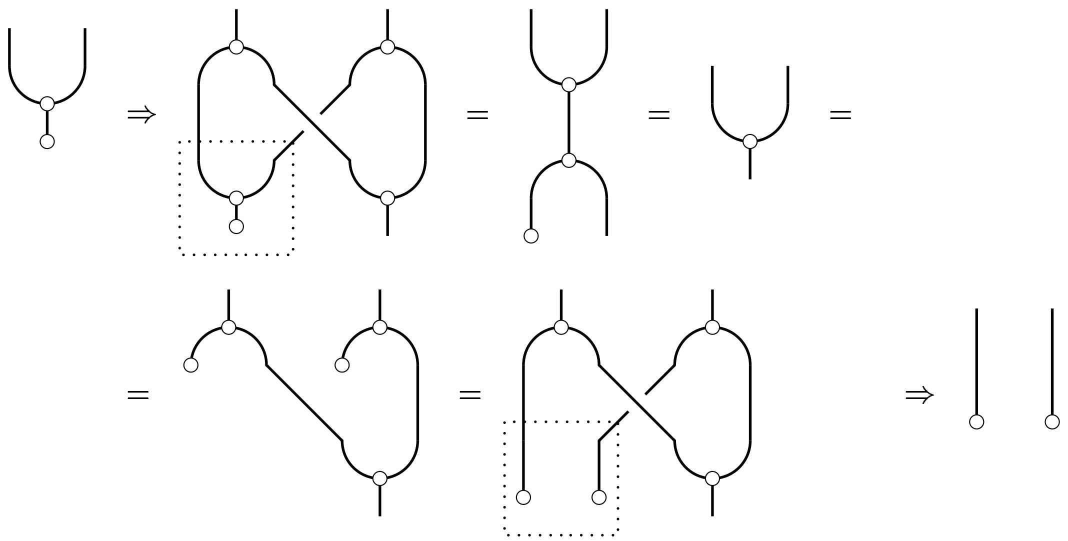
Kuperberg's lemma takes thus a central part in the theory of 3-manifold invariants and quantum field theory as promoted by Witten. Regarding our results, we will revisit this lemma. In our notation it reads

Theorem 5.10 (Kuperberg [84], Lemma 3.2). *The following identities hold in* any *Hopf algebra:*



(5-72)

Kuperberg's **proof** of a) is as follows:



(5-73)

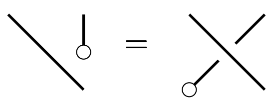
Furthermore, the proof of b) requires a), the proof of c) requires b) and the proof of d) requires c). Hence every assertion of the lemma is true if and only if a) holds.

But, we have already displayed a counterexample to property a) , see theorem 5.5! If a Hopf gebra has a non-local product, e.g. derived by cliffordization, we showed that the counit is not an algebra homomorphism, i.e. does not satisfy a). So, where is the error in the very suggestive proof?

There are two points to be criticized. First, we note that the proof uses a technique where to a tangle x a 'helping tangle' t or b is added and after some manipulations it is removed unaltered. That is this tangle acts like a catalyser in chemistry. However, if the helping tangle t is added on top it has to be monic and if the helping tangle b is added from below it has to be epi to guarantee that a cancellation law holds, i.e. the tangle can be safely removed again. Let $x = y$ be the searched for tangle equation and t from the top and b from the bottom added tangles, one computes in algebraic terms

$$\begin{aligned} x \Rightarrow \quad & xt = \ldots = yt \quad \Rightarrow \quad x = y \\ x \Rightarrow \quad & bx = \ldots = by \quad \Rightarrow \quad x = y \end{aligned} \tag{5-74}$$

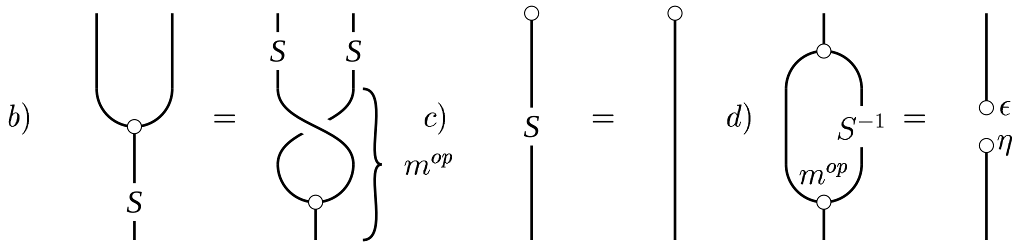
and a cancellation takes place. Now it is known [99] that the tangle $\Delta \circ m$, which occurs after the first equality sign in the proof of a), is *not* invertible. Hence we cannot assume a cancellation law to hold! Secondly, we noticed already that the crossing is defined by the structure tensors m, Δ since an antipode exists. Hence one cannot assume that



(5-75)

holds true, but this has to be proved. In fact, our counterexample, obtained by cliffordization in theorem 5.2 shows that the assertion a) is not true for *any* Hopf (al)gebra. In fact, if a Hopf algebra is *defined* to be connected, as in Sweedler or Milnor Moore, Kuperberg's assertion is evidently true, but a) is an axiom in this case and has not to be proved at all. We reformulate the Kuperberg lemma as follows:

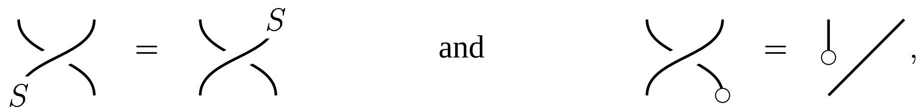
Theorem 5.11. *In any Hopf algebra where the co-product is counital and the counit is an algebra homomorphism and the product is unital, the unit is a co-product homomorphism, and Eqn. 5-77 holds then the following identities hold:*



(5-76)

Note that the product has thus to be *local* as defined in definition 5.4. In terms of the previous chapter this asserts that the algebra part of the Hopf algebra has to be an augmented connected algebra. We required the same for the coal-gebra.

The proof of $b)$ requires furthermore that



(5-77)

which are further requirements on the crossing.

5.6.2 Interacting and non-interacting Hopf gebras

The observations of the previous section lead naturally to the following questions:

Q1: Is any crossing of an antipodal convolution, i.e. Hopf gebra, which is derived from a local product and a local co-product 'knottish' in that sense, that operators can be moved on the strings under and over as done in Eqn. 5-77?

If we remember the definition of the crossing as planar graph, this is a non-trivial requirement. Looking at Graßmann Hopf gebras, one finds however that

$$\begin{aligned}(S \otimes S) \circ \hat{\tau}(a \otimes b) &= (-)^{|a||b|}(S \otimes S) \circ \mathrm{sw}(a \otimes b) \\ &= (-)^{|a||b|}\mathrm{sw} \circ (S \otimes S)(a \otimes b) = \hat{\tau} \circ (S \otimes S)(a \otimes b)\end{aligned} \tag{5-78}$$

which fulfils also

$$(f \otimes g) \circ \hat{\tau} = \hat{\tau} \circ (g \otimes f) \tag{5-79}$$

for graded morphisms f, g.

Q2: Does any Hopf gebra which possesses a crossing which fulfils the Reidemeister moves, and is thus 'knottish', has a local product and a local co-product? Or alternatively, are all Hopf gebras for 3-dimensional topology based on augmented connected algebras and connected augmented co-algebras?

Q3: Since cliffordized Hopf gebras possess non-local products, and co-cliffordized Hopf gebras possess non-local co-products, are there 'knottish' such algebras?

Q4: What is the topology behind cliffordized, i.e. quantized Hopf gebras? Is this topology related to non-commutative geometry?

Q5: Since Hopf algebras are generalizations of groups, which generalized groups are behind the non-local Hopf gebras? However, some results in this direction are available [54, 52, 51, 5, 53].

Q6: Is there a reason from physics that cliffordized, i.e. quantized, structures have to be used e.g. in quantum field theory?

We will not have occasion to answer these question in full detail in this treatise, but we will introduce a further notation which might suggest a direction towards the answers and which is motivated by demands of physics. Moreover we will show that normal-ordering is encoded this way, and a recent pre-print of Brouder [22] shows that cliffordization may be behind the curtain of Epstein-Glaser renormalization of time-ordered products.

We will see below, that in quantum field theory Hopf gebras constitute the structure of the generating functionals and of their algebraic properties. It is quite suggestive, after examining this fact, to use the counit as the vacuum expectation value [50, 55]. The below discussed topic of normal-ordering deals with the connection of local and non-local structures. This follows also from the fact, obtained in theorem 5.5, that the counit acting on the Clifford product gives the cup tangle of the scalar product. Hence this tangle replaces the cup tangle in e.g. the Kauffman bracket. The crucial fact of the property a) of the Kuperberg Lemma 3.2, as discussed above is that the counit is an algebra homomorphism. If we look at the formula in terms of an expectation value, from

$$\epsilon(ab) = \epsilon(a)\epsilon(b) \qquad\qquad \langle ab\rangle = \langle a\rangle\langle b\rangle \tag{5-80}$$

it follows that one deals with a free theory as is well known that the factorization of expectation values represents independence. A physically non-trivial theory has to have interactions between its constituting parts which renders the Hopf gebras with local products to be a less interesting case. However, it is that case which is employed, in the theory of knots and links, Kauffman bracket and Jones polynomial and therefore in Witten's approach to quantum field theory as described in [135].

This motivates the following distinction:

Definiton 5.12. *A Hopf gebra with local product and local co-product, i.e. a bilocal Hopf gebra, is called a* non-interacting *Hopf gebra, otherwise the Hopf gebra is called* interacting *Hopf gebra.*

Note that a Hopf gebra is already called interacting if *one* of the involved products, product *or* co-product is non-local. If the co-product is non-local, then the co-product has no longer the form

$$\Delta(x) = x \otimes \mathrm{Id} + \mathrm{Id} \otimes x + {\sum_{(x)}}' x_{(1)} \otimes x_{(2)} \,, \tag{5-81}$$

which is used to derive the recursive form of the antipode. The primed sum indicates proper sections of x, i.e. $x_{(i)} \neq \mathrm{Id}$. Hence, in that case the antipode formula also used by Connes and Kreimer in their renormalization theory for perturbative quantum field theory cannot be established. Nevertheless an antipode and convolutive inverse endomorphisms do exist in such cases too as we showed by direct calculation using BIGEBRA.

Chapter 6

Integrals, meet, join, unipotents, and ‘spinorial’ antipode

6.1 Integrals

We introduce a further structure in the convolution algebra, called *integral*, see e.g. Sweedler [130].

Definiton 6.1. *A* left/right integral *is an element $\mu_{L/R} \in A^*$, i.e. a (multi) covector of the unital convolution* $\mathrm{Conv}(A, A)$ *which fulfils*

(6-1)

In equation notation this reads for any x

$$(\mathrm{Id} \otimes \mu_R)\Delta(x) = \mu_R(x)\mathrm{Id} \qquad\qquad (\mu_L \otimes \mathrm{Id})\Delta(x) = \mu_L(x)\mathrm{Id}\,. \tag{6-2}$$

To distinguish integrals from unit and counit, we use black bullets in the graphical notation. Obviously zero is an integral, but a trivial one. Therefore we speak about proper or non-trivial integrals if $\mu_{R/L} \neq 0$ is non-zero.

Using duality we define analogously cointegrals.

Definiton 6.2. *A* left/right cointegral *is an element $e_{L/R} \in A$, i.e. a (multi) vector of the counital convolution* $\mathrm{Conv}(A, A)$ *which fulfils*

(6-3)

In equation notation this reads, for any $x \in \bigwedge V$

$$m(x \otimes e_R) = \epsilon(x)\, e_R \qquad\qquad m(e_L \otimes x) = \epsilon(x)\, e_L\,. \tag{6-4}$$

Of course the term integral is taken since $(\mu_L \otimes Id) \circ \Delta$ $(Id \otimes \mu_R) \circ \Delta)$ is a linear form acting in such a way that the result is a scalar and the action is linear in the argument, see Sweedler [130].

Example: (continued) We consider once more the Clifford biconvolution $\mathcal{C}\ell(B,C)$ in $\dim V = 2$, $\dim \bigwedge V = 2^2 = 4$ and B, C as defined above in the chosen basis. We find the following

Theorem 6.3. *For a Graßmann biconvolution, i.e. a Graßmann Hopf gebra, $\bigwedge V = \mathcal{C}\ell(0,0)$, i.e. $B \equiv 0 \equiv C$ identical zero, there exists one and only one non-trivial left and right integral $\mu = \epsilon^{12}$ where $\epsilon^{12}(e_{12}) = 1$ and $\epsilon^{12}(e_I) = 0$, $I \neq (12)$, and there exists one and only one non-trivial left and right cointegral $e = e_{12}$ where $e_{12}(\epsilon^{12}) = 1$ and $e_{12}(\epsilon^I) = 0$, $I \neq (12)$.*

Proof: by direct computation using CLIFFORD / BIGEBRA.

In general one finds, using physicists notation, γ^5 for the elements of maximal grade, that $\mu = \gamma^5 = \epsilon^n \vee \ldots \vee \epsilon^1$ is the *unique integral* and $e = \gamma_5 = e_1 \wedge \ldots \wedge e_n$ is the *unique cointegral* in the n-dimensional case of Graßmann Hopf gebra. One should thereby remember, that Graßmann Hopf gebras are bi-augmented and bi-connected and are thus well behaved. This situation changes drastically if we turn the products and co-products into non-local ones by cliffrodization.

Theorem 6.4. *For a Clifford biconvolution $\mathcal{C}\ell(B,C)$ ($\dim V = 2$) as defined above one obtains no non-trivial integral unless $C \equiv 0$ and no non-trivial cointegral unless $B \equiv 0$, i.e. unless the cliffordization is trivial.*

Proof: by direct computation using CLIFFORD / BIGEBRA.

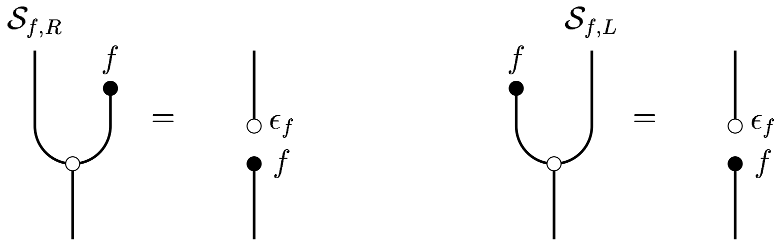
This result should be compared with various claims of the existence of integrals, e.g. see [77]. A theory of integrals for Hopf algebras having non-local products, called quasi Hopf algebras, was developed in [96, 97, 67]. Moreover, we have no doubts that these results can be generalized to arbitray dimensions which needs an algebraic proof.

The result we found above for cliffordized and thereby non-local products and co-products agrees with the fact that Graßmann algebras are faithfully represented on the module they are built over. That is, the left/right regular representation $L_a b = ab$ ($R_a b = ba$) is irreducible. This follows from the fact that Id is the only non-trivial idempotent element in $\bigwedge V$. Hence a minimal left/right ideal in $\bigwedge V$ is $\bigwedge V$ itself. This is a bi-ideal.

In the case of a cliffordized algebra one obtains new primitive idempotent elements $f_i^2 = f_i$ with Id $= \sum f_i$ and $f_i f_j = f_j f_i$. Such primitive idempotents generate left/right ideals which carry faithful representations, called *spinor representations*:

$$\begin{aligned} \mathcal{S}_f &\cong \mathcal{C}\ell\, f \\ L_{\mathcal{C}\ell}\, \mathcal{S}_f &\cong \mathcal{S}_f\,. \end{aligned} \tag{6-5}$$

Let for the moment $\Bbbk$ be the field of real number having no non-trivial involutive automorphisms. Any f_i can take over the role of the unit $\eta \;:\; \Bbbk \to A$ and $f_i\,\mathcal{C}\ell\,f_i \cong \Bbbk\,f_i$ can represent the base ring the algebra is built over. In fact f_i can take over the role of a cointegral on a left/right spinor space $\mathcal{S}_{f,L}, \mathcal{S}_{f,R}$, which is a graded $\Bbbk$-module:



(6-6)

Here we have defined the counit ϵ_f to be $f\,\mathcal{C}\ell\,f \mod f$ and the cointegral is given by f itself. Integrals could be obtained by categorial duality from this structure.

This is a tremendously important structure since it is directly related with the representation of elementary particles in quantum field theory. Moreover, the structure of the state space of a quantum system will depend strongly on this fact. One finds a decomposition of the unit $\mathrm{Id} = \sum f_i$ which induces a direct sum decomposition of the representation space, i.e. left/right ideals. Also the counit will split along the same lines as $\epsilon = \sum \epsilon_f$. We will use in this mathematical section only integrals and cointegrals of Graßmann Hopf gebras, see the discussion of meet and join below, and we will not develop a theory of integrals and cointegrals for Clifford biconvolution.

Moreover, in the physics sections below we will find that due to Wick normal-ordering the situation there is much more involved. We will discuss these peculiarities there.

6.2 Meet and join

In this section we will shortly explain in which way integrals and cointegrals are involved in Graßmann-Cayley algebras.

Starting point is a Graßmann Hopf gebra. The interpretation of 1-vectors is that of points of a projective space represented in a homogeneous way. That is, a and αa, $0 \neq \alpha \in \Bbbk$, are the same point. In fact, the field $\Bbbk$ does not play a major role in what follows, but we will assume that $\Bbbk$ is a field of characteristic 0.

The 'join' of two points a, b is the line $l = \overline{ab}$ which is represented by the exterior wedge product $\wedge$, i.e. $l = a \wedge b$. Incidence of an arbitrary point x with that line results in a linear dependence of the triple of vectors a, b, x which results in $x \wedge a \wedge b = 0$. This can be found in Graßmann [64, 63]. The incidence relation using the wedge or 'join' is a non-parametric relation. Such incidence relations have been discussed recently e.g. in Conradt [36]. It is obvious that three independent points constitute a plane etc. One notes therefore that the exterior wedge or 'join' product raises the grade and increases the dimension of the geometric objects represented by them.

The dual question to the join is that of a 'meet'. Two lines $l = \overline{ab}$ and $m = \overline{cd}$, represented by arbitrary points a, b and c, d incident with them, meet eventually in a point x or not. The meet is represented by the exterior vee-product $\vee$, e.g. $l \vee m$.

Using duality between points and planes in $\mathbb{P}^3$, one finds that the meet constitutes an exterior vee-algebra $\bigvee V^*$ of planes. That is, the meet of 2 planes is a line and the meet of 3 planes is a point. Hence the meet increases the grade in the space constituted from planes. It lowers the grade if planes and lines are seen to be represented by sets of points.

Graßmann introduced the meet in the **A2** by means of an Ergänzungs operator. This *Ergänzung* is related to an orthogonal complement and denoted by a vertical bar $|$. Graßmann defined it by an implicit relation

$$[a \wedge |a] = 1 \tag{6-7}$$

where $|a$ is the Ergänzung and $[\ldots]$ is a volume form as studied in the case of Peano space. In fact, that is Peano's source [105].

The meet, alias regressive product or 'eingewandtes Produkt', as opposed to the exterior (progressive) product, was defined by Graßmann [63] as

$$|(a \vee b) := (|a) \wedge (|b)\,. \tag{6-8}$$

This relation is still projective and does not use a metric but depends on a symmetric correlation. One should compare this Graßmannian definition with that of Hestenes, Sobczyk [69] and Hestenes, Ziegler [70] where inner products are used. This route was taken also by Conradt [36, 38].

Recall that the bracket $[.,\ldots,.] \;:\; \otimes^n V \to \Bbbk$ was essentially identical to a determinant of the matrix of the vector components of its entries

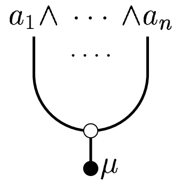
$$\det(a_1,\ldots,a_n) = [a_1,\ldots,a_n]\,. \tag{6-9}$$

But following Chevalley [31], the determinant can be calculated along the following line. Let $a_i = A(e_i)$ be the images of some e_i, which constitute a basis of V, i.e. $[e_1,\ldots,e_n] \neq 0$. Let $A^\wedge$ be the graded extension of A on $\bigwedge V$, as we have extended the scalar and coscalar products above. One computes

$$\begin{aligned} A^\wedge(e_1 \wedge \ldots \wedge e_n) &= a_1 \wedge \ldots \wedge a_n = \alpha\, e_1 \wedge \ldots \wedge e_n \\ \det(A) &= \alpha = [a_1,\ldots,a_n]\,. \end{aligned} \tag{6-10}$$

Not even an orientation is needed, the determinant respects only a relative orientation between two sequences here. But this will change if a particular basis is selected and orientation is established relatively to such a *right handed basis*. Now we find that the cointegral of a Graßmann Hopf gebra projects onto the highest grade element, hence on the determinant of that linear operator, which maps a certain basis fulfilling $[e_1,\ldots,e_n] = 1$, i.e. linearily ordered and oriented, onto the input of the bracket. We *define* the bracket using the unique Graßmann Hopf integral

$$[a_1, \ldots, a_n]_\mu \quad \cong$$

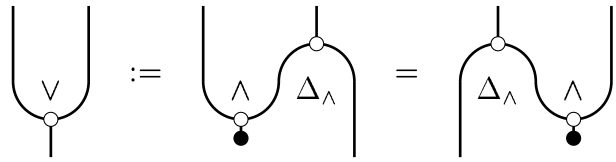


(6-11)

The Hopf algebraic definition of the meet as given by Doubilet, Rota and Stein [43] reads

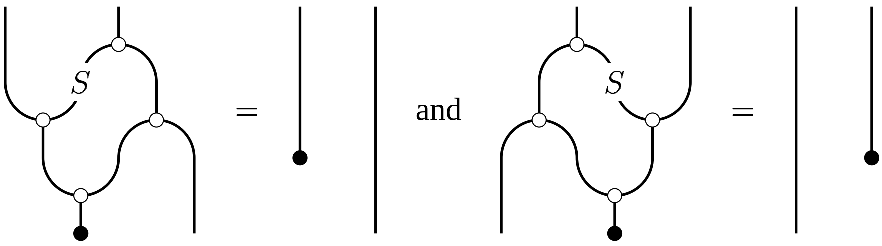
$$\begin{aligned} a \vee b &= (a_1 \wedge \ldots \wedge a_r) \vee (b_1 \wedge \ldots \wedge b_s) \\ &= [b_{(1)}, a]\, b_{(2)} = a_{(1)}\, [b, a_{(2)}] \\ &= \pm [a, b_{(1)}]\, b_{(2)} = \pm a_{(1)}\, [a_{(2)}, b] \end{aligned} \tag{6-12}$$

and contains the bracket. That the bracket is not foreign to the Graßmann Hopf gebra was discussed in the previous section. However, we can now see that the bracket involves a disguised integral. In the following tangles we use for clarity the last line for the meet and compute modulo signs, which is allowed in homogeneous coordinates of projective geometry. We define hence the meet as



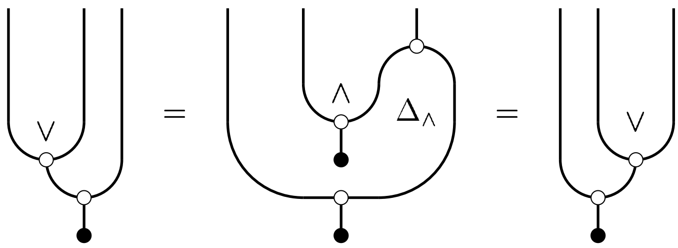
(6-13)

A short calculation shows that the meet $\vee$ is associative. Note that the r.h.s. consists of Kuperberg ladder diagrams truncated by the cointegral. The second equality was proved by Doubilet, Rota and Stein [43]. Since the Kuperberg ladder tangles are invertible, we can derive the relations



(6-14)

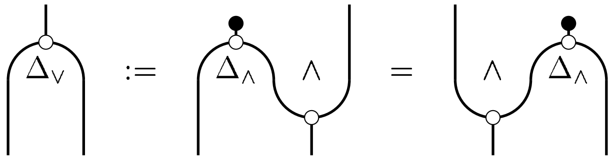
Having the $2 \to 0$ tangle for the bracket, it is natural to introduce the product co-product duality w.r.t. this cup tangle:



$$\mu((a \vee b) \wedge c) \;=\; \mu(a \wedge (b \vee c))\,, \tag{6-15}$$

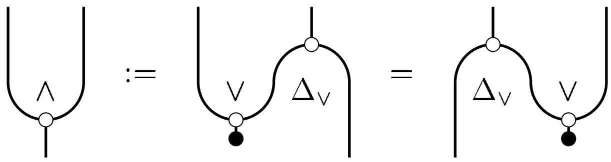
which is a straigthening formula also derived by Rota et al. In this particular sense, $\vee$ is a self dual product.

We know already from the Graßmann Hopf gebra that by categorial duality we can derive a Graßmann co-product from the exterior wedge product $\wedge \rightarrow \Delta_\wedge$. Along the same lines, i.e. using eval, we can introduce a Graßmann co-product for the meet $\vee \rightarrow \Delta_\vee$. This notion depends on the unique integral of the Graßmann Hopf gebra



(6-16)

The full symmetry of this structure was already noted by A. Lotze 1955 [86], using Hopf gebras only implicitly in the combinatorics of indices. Lotze showed that the exterior product derived from the meet along the same lines as the meet itself was obtained, is again the join!



(6-17)

Denoting the Ergänzung in modern notation by a star $\star$, the full mathematical structure turns out to be the *Graßmann-Cayley double Hopf gebra* or *fourfold algebra*

$$GC(\bigwedge V, \bigvee V^*, \wedge, \Delta_\wedge, \vee, \Delta_\vee, \star) \tag{6-18}$$

which possesses also units, counits, antipodes $S^\wedge$, $S^\vee$, integrals and cointegrals. Furthermore this structure is the vector space analog of the Boolean algebra of sets, the algebra of logical inference.

One can check by easy computations that the cointegral is the unit of the meet while the integral is the unit of the meet co-product. The integrals and cointegrals obtained from the meet play the same role for the wedge again.

Unfortunately we have no further opportunity to discuss the geometry behind this interesting highly symmetric algebraic structure in this work.

6.3 Crossings

We examine some properties of the crossing derived from the antipode. This will be done for our $\dim V = 2$ example. This is not a theory of the crossing, but it gives valuable hints how the crossing behaves.

Example: (continued) Let $\dim V = 2$ and B, C be the arbitrary scalar and co-scalar products of the Clifford biconvolution as in the previously discussed cases.

Theorem 6.5. *The Clifford biconvolution $\mathcal{C}\ell(B,C)$ is commutative as an algebra*

$$m \circ \hat{\tau} = m \tag{6-19}$$

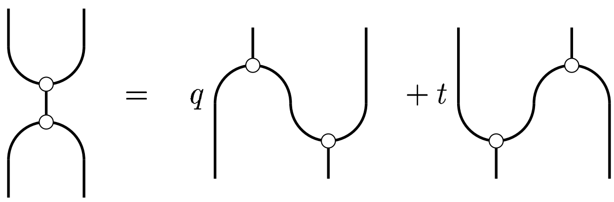
if the scalar product is identically zero, $B \equiv 0$, and the co-scalar product is symmetric, $C^T = C$. The Clifford bi-convolution $\mathcal{C}\ell(B,C)$ is commutative as a co-gebra

$$\hat{\tau} \circ \Delta = \Delta \tag{6-20}$$

if the co-scalar product is identically zero, $C \equiv 0$, and the scalarproduct is symmetric, $B^T = B$.

Proof: by direct computation using CLIFFORD / BIGEBRA.

We will furthermore check if we can derive a non-knottish skein relation for the crossing, as various such relations have been suggested by Oziewicz. Let $\dim V = 2$ and $\mathcal{C}\ell(B,C)$ be an arbitrary Clifford biconvolution. Are there solutions to the following skein relation, where the Kuperberg ladders are involved?



(6-21)

As a result we obtain pairs of scalar and coscalar products (B,C) such that the skein relation holds true for $q = 0$ or $t = 0$. However, we find also solutions of the form

$$\begin{aligned}
&q = 1 - t \qquad && B = \begin{bmatrix} \frac{1}{u} & -\frac{z}{uv} \\ 0 & \frac{1}{v} \end{bmatrix} \qquad && C = \begin{bmatrix} u & z \\ 0 & v \end{bmatrix} \\
&q = 1 - t \qquad && B = \begin{bmatrix} -\frac{vc}{w} & \frac{vuc+w}{w^2} \\ c & -\frac{uc}{w} \end{bmatrix} \qquad && C = \begin{bmatrix} u & \frac{vuc+w}{wc} \\ w & v \end{bmatrix} .
\end{aligned} \tag{6-22}$$

It would be an interesting task to investigate which sort of skein relations can appear in this way and in which way such skein relations can be used in physics. However we will follow another route here.

6.4 Convolutive unipotents

An unipotent element x of an unital algebra fulfils $x^2 = \mathrm{Id}$. Idempotent elements which square to themselves $f^2 = f$, are related to unipotents. That is, from every non-trivial unipotent element x one can construct two idempotent elements which are mutually orthogonal

$$f_\pm = \frac{1}{2}(\mathrm{Id} \pm x) \qquad \Rightarrow \qquad f_\pm^2 = f_\pm, \qquad f_+ f_- = f_- f_+ . \tag{6-23}$$

The knowledge of commuting unipotents is thus closely related to that of (primitive) idempotents. But primitive idempotents generate minimal left and right ideals which carry irreducible faithful representations of algebras, if this algebra is fully reducible.

In a Graßmann algebra all elements but Id are nilpotent, i.e. $x^n = 0$ for some $n \in \mathbb{Z}$. Hence Id and 0 are the only unipotent elements which are at the same time idempotents. That is, $\bigwedge V$ is faithfully and irreducibly represented on $\bigwedge V$ seen as $\mathbb{k}$-module.

Cliffordization changes this fact drastically. The non-local Clifford product generates, depending on the base field and the involved bilinear form, a certain number r of commuting unipotents. From these r commuting unipotents, 2^r primitive idempotents are constructed. The number r is the Radon-Hurwitz number found by Hurwitz and Radon during their studies of the composition of quadratic forms [73, 114]. Benno Eckmann showed that there is a group theoretical root of this number [45] and provided a short proof. Later, Hasler Whitney showed, that the Radon-Hurwitz number is related to the number of independent vector fields on spheres [133]. That is, the Radon-Hurwitz number is directly related to topological properties of the underlying group manifolds. This provides a relation to the topological relation of Hopf algebras and the process of cliffordization of Hopf gebras.

Since we saw that the convolution establishes a group like structure via the convolution and finally the Hopf gebra. We can now start to study the endormorphsims of $\bigwedge V$ forming a convolution Hopf algebra and ask if there are convolutive unipotents in Hopf gebras.

Theorem 6.6. *In a Graßmann Hopf gebra the convolutive unit u is up to the sign the only unipotent element.*

Hence the unit and zero $(u, 0)$ are the only idempotent endomorphisms under the Graßmann convolution product. That is, we expect the totality of all graded endomorphisms $\mathrm{End} \bigwedge V$ to form an irreducible representation space for the convolution as was the case for the Graßmann algebra itself. Once more cliffordization changes the situation.
Example: (continued) We consider the $\dim V = 2$ Clifford bi-convolution $\mathcal{C}\ell(B, C)$.

Theorem 6.7. *There are more than 90 non-trivial $T \neq 0$ convolution unipotent solutions of the equation $T \star T = u$ in the Clifford biconvolution $\mathcal{C}\ell(B, C)$ of $\dim V = 2$. [Not all of them being independent.] In particular, among the solutions there are singular endomorphisms $\det T = 0$ and non-singular endomorphisms $\det T \neq 0$.*

Proof: by direct computation using CLIFFORD / BIGEBRA.

We expect thus a non-trivial and highly interesting representation theory of Clifford biconvolutions and Clifford Hopf gebras. However, we cannot enter this topic here.

6.4.1 Convolutive 'adjoint'

The notion of an adjoint operator belongs to the theory of inner product spaces. One defines the adjoint operator A^* to be the operator shifted into the left slot of the inner product

$$\langle xA^* \mid y \rangle = \langle x \mid Ay \rangle \,. \tag{6-24}$$

A certain type of ’adjointness’ is the inverse which is defined w.r.t. a product, not a bilinear form. In the case of a group one has

$$g \star h = \mathrm{Id} \star g^{-1}h = gh^{-1} \star \mathrm{Id}\,. \tag{6-25}$$

This gives an idea to introduce a *convolutive adjoint*, resp. inverse, along the lines of the group inversion. However, in the case of groups the identity of Eqn. 6-25 is trivial, since $\star$ is the group multiplication identical to juxtaposition. But in the case of a Hopf gebra or a convolution algebra, the convolution product is *not* identical with the repeated application or composition of endomorphisms:

$$\begin{array}{rl} \text{group} & g \star g^{-1} = \mathrm{Id} = gg^{-1} \\ \text{conv. alg.} & g \star g^{-1} = \mathrm{Id} \star g^{\star}g^{-1} = g^{-1}g^{\star} \star \mathrm{Id}\,. \end{array} \tag{6-26}$$

6.4.2 A square root of the antipode

The convolutive inverse is mediated by the antipode. However, we can use the above found unipotents which are related to the antipode itself by convolutive adjointness

$$\begin{array}{rl} & T \star T = u \\ \Rightarrow & \mathrm{Id} \star T^{\star}T = T\,T^{\star} \star \mathrm{Id} = u\,. \end{array} \tag{6-27}$$

The second line is the defining relation for the antipode. Hence we find

$$T^{\star}T = S = TT^{\star} \tag{6-28}$$

or equivalently

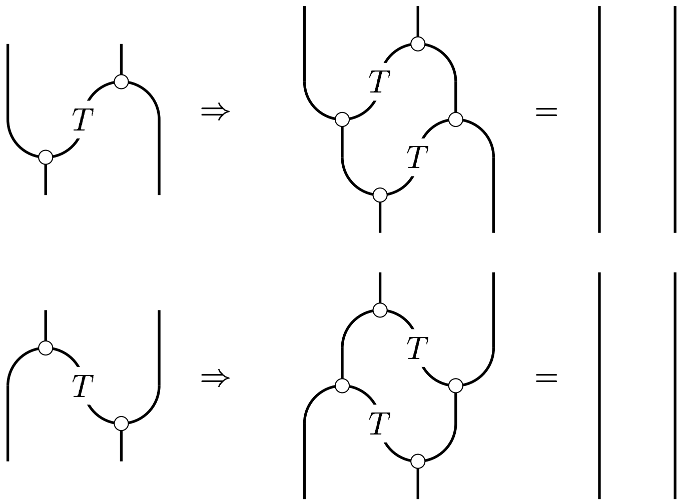
$$S(T)T = S = TS(T)\,. \tag{6-29}$$

This mechanism is related to Möbius inversion of polynoms, which we unfortunately cannot examine here. However, we can report that using BIGEBRA we have been able to find a great variety of operators T, which proves their existence in some special cases. Moreover, we find invertible and singular Ts which will induce a rich representation theory.

In a certain sense, the operator T is like a square root of S and could be called *spinorial antipode* since it parallels by analogy the $\mathrm{spin} \rightarrow SO$ or $\mathrm{pin} \rightarrow O$ covering. It is not yet clear, if such operators are connected to coverings of the topological spaces behind the Hopf algebra.

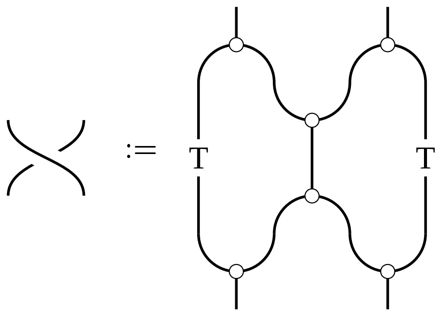
6.4.3 Symmetrized product co-procduct tangle

As an application, we can symmetrize the Kuperberg ladder tangles. Using the convolution unipotent T, we define



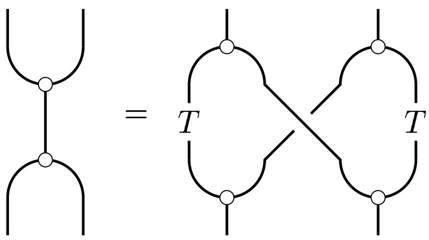
(6-30)

That is, these tangles are also unipotents. However, our choice implies further alterations in the theory. If we stay with the crossing of Oziewicz, but change $S \to T$, we get



(6-31)

which leads to a deformation of the crossed products



(6-32)

If this is not asserted, we can no longer prove from the crossing that Δ and m are algebra and cogebra morphisms, which is however a main feature of the Hopf gebra.

A detailed study of the representation theory of convolution algebras will provide further information in which way such a generalization has meaningfully to be developed.

Chapter 7

Generalized cliffordization

Rota and Stein developed indeed a much more general concept of cliffordization [119] as we have till now used. They used a product like mapping $\&r : \bigwedge V \times \bigwedge V \to \bigwedge V$ with a non-scalar target as the 'cup' tangle in the cliffordization. Such a tangle is no longer a 'sausage', but has a third line which is internal and downwards. However, it is not clear why a product should be deformed by another *a priori* given product. However, quantum mechanics uses complex valued (anti)commutators which are maps $[.,.] : V \times V \to \Bbbk$. If the target is extended to $\bigwedge V$ then the (anti)commutator products are no longer complex valued but operator valued. Since the (anti)commutator algebra emerges itself from a cliffordization, as demonstrated above, this would imply a cliffordization of a cliffordized product, hence a cliffordization of second order. In terms of tangles a sausage tangle inside a second more general cliffordization. While this is interesting in its own right, we will generalize cliffordization to the most general case where the cup tangle is scalar valued, i.e. where the tangle remains to be a sausage. It will turn out, using a result of Brouder [22], who used Hopf algebraic formulas of Pinter [111, 110], that a scalar valued bilinear form may be already sufficient to be able to introduce renormalized time-ordered operator products and correlation functions as required by Epstein-Glaser renormalization [46]. This mechanism introduces the renormalization parameters via a scalar valued Z-pairing. While this is a special bilinear form, i.e. cup tangle, we consider in this chapter the general case. This will allow us to derive some of the defining relations of the Z-pairing of renormalization by reasonable assumptions about the resulting cliffordized product.

7.1 Linear forms on $\bigwedge V \times \bigwedge V$

In the preceeding section we were interested in generalizing Clifford algebras of a quadratic form $\mathcal{C}\ell(V,g)$ to quantum Clifford algebras of an arbitrary bilinear form $\mathcal{C}\ell(V,B)$. The form

$$B : V \times V \to \Bbbk \tag{7-1}$$

is defined on V and has n^2 independent parameters. This form has to be extended to a bilinear form

$$B^\wedge : \bigwedge V \times \bigwedge V \to \Bbbk \tag{7-2}$$

which obviously has also only n^2 parameters. However, a general bilinear form $\mathcal{BF} : \bigwedge V \times \bigwedge V \to \Bbbk$ has $2^n \times 2^n = 4^n$ parameters. The aim of this chapter is to investigate what kind of restrictions follows for $\mathcal{BF}$ if we assert some properties to the cliffordized product based on this bilinear form.

First of all, we recall what kind of bilinear form we have used till now. The bilinear form $B^\wedge$, which we defined by exponentiation or as a Laplace pairing, is graded in the following sense (on two extensors and extended by bilinearity):

$$B^\wedge(x_1 \wedge \ldots \wedge x_r, y_1 \wedge \ldots y_s) = \begin{cases} B(\mathrm{Id}, \mathrm{Id}) & \text{if } r = 0 = s \\ B(x, y) & \text{if } r = 1 = s \\ (-)^{\frac{r(r-1)}{2}} \det([B(x_i, y_j)]) & \text{if } r = s \\ 0 & \text{otherwise.} \end{cases} \tag{7-3}$$

Note that we have used a different indexing in the first factor as previously, which results in the prefactor in front of the determinants due to reversion of factors. In a standard basis this yields a matrix representation where $B^\wedge$ decomposes into a block structure

$$[B^\wedge] = \begin{bmatrix} 1 & 0 & \ldots & \ldots & \ldots \\ 0 & B & 0 & \ldots & \ldots \\ \vdots & 0 & B^2 & 0 & \ldots \\ \vdots & \vdots & 0 & \ddots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{7-4}$$

where $B^r : \bigwedge^r V \times \bigwedge^r V \to \Bbbk$ is an $\binom{r}{n} \times \binom{r}{n}$-matrix and a function of B. The grading enforces the off-block entries to be zero. One sees easily that the B^r's are built from minors of the original bilinear form B.

Example: Let $\dim V = 3$ and B be represented in an arbitrary basis $\{e_i\}$ as a 3×3-matrix. B^0 is defined to have the value 1. This requirement will allow the unit of an augmented connected Hopf gebra to stay to be the unit w.r.t. the cliffordized product, i.e. $\mathrm{Id} \,\&_c\, \mathrm{Id} = B(\mathrm{Id}, \mathrm{Id})\mathrm{Id} = \mathrm{Id}$. B^1 is obviously identical to B. An entry of B^2 is given as

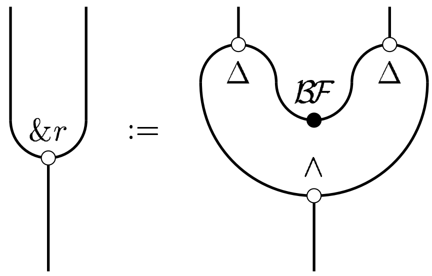
$$B^2(e_i \wedge e_j, e_k \wedge e_l) = (-)^{2(2-1)/2}(B(e_i, e_k)B(e_i, e_l) - B(e_j, e_l)B(e_i, e_k)) \tag{7-5}$$

and all of them make up a 3×3-matrix, while $B^3 = -det([B(e_i, e_j)])$ is the determinant of B. Taking the trace of $B^\wedge$, one obtains the invariants of the bilinear form B, which could be reformulated using the eigenvalues λ_i of B as $\{1, \sum \lambda_i, \sum_{i<j} \lambda_i\lambda_j, \sum_{i<j<k} \lambda_i\lambda_j\lambda_k = \prod \lambda_i\}$.

A first generalization would break up this correlation of the $\binom{n}{r} \times \binom{r}{n}$-block-matrices arising from B. While B has n^2 parameters, such a general graded bilinear form would come up with $\sum_r \binom{n}{r} = 2^n$ parameters. If we would introduce a grade operator (particle number operator) such maps would commute.

However, it is at this stage more convenient to introduce an ungraded bilinear form $\mathcal{BF} : \bigwedge V \times \bigwedge V \to \Bbbk$ without any restrictions having 4^n free parameters. Alternatively this map can be regarded as a *linear form* on the space $\bigwedge V \times \bigwedge V$. The tangle does not change at all and remains to be a cup tangle. However, the properties of the product obtained by cliffordization, and hence the (Hopf) algebra built from it change *dramatically*! In fact this generalization will be sufficient to allow us to incorporate renormalization directly into a cliffordization process. While we showed in [56] that time- and normal-ordered correlation functions and operator products are related by a cliffordization process, C. Brouder [22] noted that Epstein-Glaser renormalization [46] may be incorporated into this process too. It was the achievement of Pinter [111, 110] to prove the equivalence of Epstein-Glaser renormalization, which resides in position space, to the Connes-Kreimer renormalization [81, 82, 34, 35] which is equivalent to the BPHZ renormalization in momentum space and the forest formulas there.

We define the *generalized Clifford product*



(7-6)

as in Rota-Stein [119], but now for a general cup tangle. Algebraically this is equivalent to

$$x \,\&r\, y = \mathcal{BF}(x_{(2)}, y_{(1)}) x_{(1)} \wedge y_{(2)}. \tag{7-7}$$

This product is called a 'generalized' Clifford product, since it leads to algebras which are Clifford like, but different to classical Clifford algebras. We agree to call this Clifford algebras also quantum Clifford algebras (QCA).

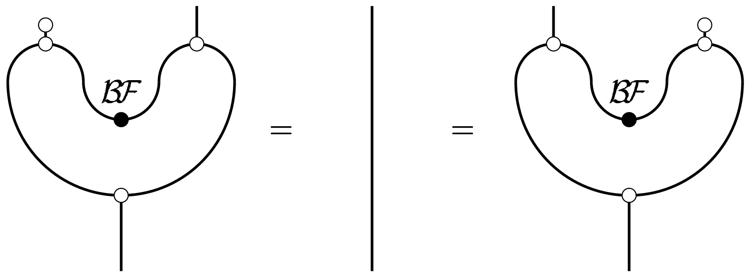
7.2 Properties of generalized Clifford products

We have unfortunately not the opportunity to develop a theory of generalized Clifford products, so we concentrate on some essential properties which we want to assert on the product emerging from cliffordization to be able to utilize it in quantum field theory. Such properties of the product will have a direct impact on the possible form of the bilinear form $\mathcal{BF}$. Our original wedge products were associative and unital. We will use these wedges to model operator products in QFT. Since we are interested especially in renormalized time-ordered operator products inside of

Wick monomials and correlation functions, we will study associative unital generalized Clifford products. These restrictions enforced on the generalized Clifford product $\&r$ will allow us to derive the assertions made on the renormalization parameters and the Z-grading by Brouder as a direct consequence.

7.2.1 Units for generalized Clifford products

We can firstly ask under which conditions the unit of the Graßmann algebra remains to be the unit of the cliffordized product $\&r$. Hence we have the condition



(7-8)

Because the co-product of the Graßmann Hopf gebra is connected and augmented, the unit is a co-algebra homomorphism as discussed above. In formulas: $\Delta \circ \eta = \eta \otimes \eta$. The same holds true for the counit. The condition on $\mathcal{BF}$ which asserts that η becomes the unit w.r.t. $\&r$ reads then

(7-9)

or in formulas

$$\mathcal{BF}(\eta, X) = \epsilon(X) = \mathcal{BF}(X, \eta) \qquad \forall X \in \bigwedge V. \tag{7-10}$$

Using a matrix representation, this implies the following block structure for $\mathcal{BF}$

$$[\mathcal{BF}] = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots \\ 0 & B_{1,1} & \dots & B_{1,n} & \dots \\ \vdots & \vdots & \ddots & \dots & \dots \\ 0 & B_{n,1} & \vdots & B_{n,n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{7-11}$$

which is a modest restriction. The renormalization scheme of Brouder-Epstein-Glaser (BEG) fulfils exactly this requirement.

However, it is an incidence and not an automatism that the Graßmann unit Id remains to be the algebra unit under the generalized Clifford product. An algebra homomorphism maps only unit onto unit and has not to have the unit as an invariant element. In fact very general elements of $\bigwedge V$ can be made to be a left/right unit w.r.t. the new Clifford product $\&r$ by a suitable choice of the bilinear form $\mathcal{BF}$.

Let $GB = \{\mathrm{Id}, e_1, \ldots, e_n, e_1 \wedge e_2, \ldots\}$ be a canonical Graßmann basis of $\bigwedge V$. An element

$$X = X_0 \mathrm{Id} + \sum_i X_i e_i + \sum_{i<j} X_{ij} e_i \wedge e_j + \ldots X_{1\ldots n} e_1 \wedge \ldots \wedge e_n \tag{7-12}$$

where some or all $X_{i_r,\ldots,i_s}$ are non-zero is a left unit if the corresponding rows of $\mathcal{BF}$ fulfil some constraints.

Example: We consider a 3-dimensional space V, $\dim \bigwedge V = 2^3 = 8$ and obtain using BIGEBRA the following matrix representation of $\mathcal{BF}$ if we assert that

$$X = X_0 \mathrm{Id} + \sum X_i e_i \tag{7-13}$$

is the left unit of $\&r$

$$[\mathcal{BF}] = \begin{bmatrix} \frac{1}{X_0} & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & 0 \\ B_{4,1} & \ldots & \ldots & B_{4,8} \\ \vdots & \ldots & \ldots & \vdots \\ B_{8,1} & \ldots & \ldots & B_{8,8} \end{bmatrix} \tag{7-14}$$

This shows that one can have non-trivial such units.

The most interesting case is however given by an element X which is a primitive idempotent element. Of course, this requires the basic product and/or co-product to be non-local and we deal no longer with an augmented connected Hopf gebra, which makes this construction peculiar. But, such an element can easily be turned into a left or right unit by the above mechanism. Idempotent elements are connected with minimal left ideals and 'spinor' representations. Probably more important is the fact, that the same process can be established for counits. However, counits can be related to the vacuum structure of a QFT. Generalized Clifford co-products allow hence a great variety of possible candidates for vacua. However, we will see that other demands restrict $\mathcal{BF}$ further.

7.2.2 Associativity of generalized Clifford products

Since renormalized time-ordered products are required to remain to be associative, we ask next which conditions are necessary for $\mathcal{BF}$ to ensure associativity of $\&r$. It will turn out, that this is a quite strong restriction and that one might be forced to drop associativity. Recall that the definition of $\&r$ was

$$u \,\&r\, v = \mathcal{BF}(u_{(2)}, v_{(1)}) u_{(1)} \wedge v_{(2)} \tag{7-15}$$

with an arbitrary bilinear form $\mathcal{BF}$, $\&r$ may not even be unital. We need further more that the co-product Δ is an algebra homomorphism

$$\begin{aligned}\Delta(a \wedge b) &= (a \wedge b)_{(1)} \otimes (a \wedge b)_{(2)} \\ &= (-1)^{|a_{(2)}||b_{(1)}|}(a_{(1)} \wedge b_{(1)}) \otimes (a_{(2)} \wedge b_{(2)}).\end{aligned} \tag{7-16}$$

Compute

$$\begin{aligned}(u \,\&r\, v) \,\&r\, w &= \mathcal{BF}(u_{(2)}, v_{(1)})(u_{(1)} \wedge v_{(2)}) \,\&r\, w \\ &= (-1)^{|u_{(12)}||v_{(21)}|}\mathcal{BF}(u_{(2)}, v_{(1)})\mathcal{BF}(u_{(12)} \wedge v_{(22)}, w_{(1)})(u_{(11)} \wedge v_{(21)} \wedge w_{(2)}) \\ &= (-1)^{|u_{(21)}||v_{(2)}|}\mathcal{BF}(u_{(22)}, v_{(11)})\mathcal{BF}(u_{(21)} \wedge v_{(12)}, w_{(1)})(u_{(1)} \wedge v_{(2)} \wedge w_{(2)})\end{aligned} \tag{7-17}$$

where we have used co-associativity and (graded) co-commutativity of the Graßmann co-product, which results in the replacements $(u_{(11)}, u_{(12)}, u_{(2)}) \to (u_{(1)}, u_{(21)}, u_{(22)})$, $(v_{(1)}, v_{(21)}, v_{(22)}) \to (v_{(11)}, v_{(12)}, v_{(2)})$ and $v_{(22)} \otimes v_{(21)} = (-1)^{|v_{(22)}||v_{(21)}|} v_{(21)} \otimes v_{(22)}$. In the same manner we compute

$$\begin{aligned}u \,\&r\, (v \,\&r\, w) &= u \,\&r\, (\mathcal{BF}(v_{(2)}, w_{(1)})(v_{(1)} \wedge w_{(2)})) \\ &= (-1)^{|v_{(12)}||w_{(21)}|}\mathcal{BF}(u_{(2)}, v_{(11)} \wedge w_{(21)})\mathcal{BF}(v_{(2)}, w_{(1)})(u_{(1)} \wedge v_{(12)} \wedge w_{(22)}) \\ &= (-1)^{|v_{(2)}||w_{(22)}|}\mathcal{BF}(u_{(2)}, v_{(11)} \wedge w_{(12)})\mathcal{BF}(v_{(12)}, w_{(11)})(u_{(1)} \wedge v_{(2)} \wedge w_{(2)})\end{aligned} \tag{7-18}$$

where once more co-associativity and co-commutativity have been used. The requirement of associativity

$$(u \,\&r\, v) \,\&r\, w = u \,\&r\, (v \,\&r\, w) \tag{7-19}$$

implies that the coefficients of the above equations have, after a renaming, to fulfil

$$(-1)^{|u_{(1)}|}\mathcal{BF}(u_{(2)}, v_{(1)})\, \mathcal{BF}(u_{(1)} \wedge v_{(2)}, w) = (-1)^{|w_{(2)}|}\mathcal{BF}(u, v_{(1)} \wedge w_{(2)})\, \mathcal{BF}(v_{(2)}, w_{(1)}) \tag{7-20}$$

which can be rewritten using the antipode as

$$\mathcal{BF}(u_{(2)}, v_{(1)})\, \mathcal{BF}(S(u_{(1)}) \wedge v_{(2)}, w) = \mathcal{BF}(u, v_{(1)} \wedge S(w_{(2)}))\, \mathcal{BF}(v_{(2)}, w_{(1)}). \tag{7-21}$$

This requirement should be compared with Brouder's coupling identities [22]. The coupling identity of Brouder is close but not identical to the conditions given by Rota-Stein for a Laplace Hopf algebra [119].

We can try to simplify further this condition by employing product co-product duality. That is, we can Laplace expand the bilinear forms having a wedge product in one of its arguments

$$\mathcal{BF}(u, v \wedge w) = \mathcal{BF}(u_{(1)}, w)\mathcal{BF}(u_{(2)}, v). \tag{7-22}$$

This yields

$$(-1)^{|u_{(1)}|}\mathcal{BF}(u_{(2)},v_{(1)})\,\mathcal{BF}(u_{(1)},w_{(2)})\,\mathcal{BF}(v_{(2)},w_{(1)}) = \\ (-1)^{|w_{(2)}|}\mathcal{BF}(u,v_{(1)}\wedge w_{(2)})\,\mathcal{BF}(v_{(2)},w_{(1)}). \tag{7-23}$$

Cancelling out the common factor and renaming yield the product co-product duality up to a sign. We get

$$\mathcal{BF}(u,v\wedge w) = (-1)^{|w||u_{(1)}|}\mathcal{BF}(u_{(1)},w)\,\mathcal{BF}(u_{(2)},v). \tag{7-24}$$

If $\mathcal{BF}$ is a graded bilinear form, we have shown

Theorem 7.1. *A graded pairing which obeys product co-product duality results in an associative cliffordization.*

On the other hand, if we assume that Id remains to be a unit we find additionally the following set of relations

$$\begin{aligned}\mathcal{BF}(\mathrm{Id},\mathrm{Id}) &= 1\\ \mathcal{BF}(\mathrm{Id},X) &= \epsilon(X)\\ \mathcal{BF}(X,\mathrm{Id}) &= \epsilon(X)\end{aligned} \tag{7-25}$$

which can be completed by product co-product duality and the requirement that $\mathcal{BF}^{-1}$ is the convolutional inverse of $\mathcal{BF}$, in formulas

$$\epsilon(u)\epsilon(v) = \mathcal{BF}(u_{(1)},v_{(2)})\,\mathcal{BF}^{-1}(u_{(2)},v_{(1)}) \tag{7-26}$$

which results in

$$\begin{aligned}\mathcal{BF}^{-1}(u,v) &= \mathcal{BF}(S(u),v)\\ \mathcal{BF}^{-1}(u,v) &= \mathcal{BF}(u,S(v))\\ \mathcal{BF}(u,v) &= \mathcal{BF}(S(u),S(v))\end{aligned} \tag{7-27}$$

where we have assumed that $S^2 = \mathrm{Id}$ is involutive. Such a structure is called a co-quasitriangular structure [91]. This structure will be investigated elsewhere. The above condition derived from associativity may be addressed as a 2-cocycle, even if product co-product duality does not hold.

7.2.3 Commutation relations and generalized Clifford products

Quantum field theory needs not only Wick monomials, normal- and time-ordered products and correlation functions, but employs also a canonical field quantization. Since the quantization encodes properties of the system under consideration they should not be altered by renormalization.

Therefore, since we want to go for renormalization, one has to assert that (anti)commutation relations are not altered by the generalized cliffordization process and thus by the renormalization process. But, this is true only for the formulation of the commutation relations on $V \times V$, higher terms might be altered by renormalization effects!

We investigate what kind of assertion is required on $\mathcal{BF}$ to guarantee that the basic (anti)-commutation relations of the generators (field operators) remain unaltered. We have to demand

$$\begin{aligned} \{\psi_1, \psi_2\}_+ &= 2\, g_{12} \qquad \text{or equivalently} \\ \{e_1, e_2\}_+ &= 2\, g_{12} \end{aligned} \tag{7-28}$$

where g is extended as usual to $\bigwedge V$ by exponentiation which yields $g^\wedge$. However, replacing the $\wedge$ in the anticommutator by the generalized Clifford product $\&r$ one ends up with additional terms. Note that

$$\begin{aligned} e_i \,\&r\, e_j &= \mathcal{BF}(e_{i(2)}, e_{j(1)}) e_{i(1)} \wedge e_{j(2)} \\ &= \mathcal{BF}(\mathrm{Id}, \mathrm{Id}) e_i \wedge e_j + \mathcal{BF}(\mathrm{Id}, e_j) e_i + \mathcal{BF}(e_i, \mathrm{Id}) e_j + \mathcal{BF}(e_i, e_j)\mathrm{Id} \end{aligned} \tag{7-29}$$

which comes up with the unwanted second and third terms. Hence our demand is that

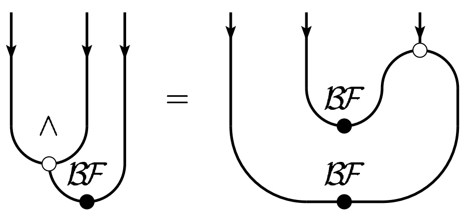
$$\begin{aligned} \mathcal{BF}(\mathrm{Id}, a) &= 0 \qquad \forall a \in V \\ \mathcal{BF}(a, \mathrm{Id}) &= 0 \qquad \forall a \in V. \end{aligned} \tag{7-30}$$

It turns out, that this is the case in the definitions of the pairings for renormalization. We will see later, that the Z-pairing has to be a $\mathbb{Z}_2$-graded morphism, which gives a second argument for this relation.

7.2.4 Laplace expansion i.e. product co-product duality implies exponentially generated bilinear forms

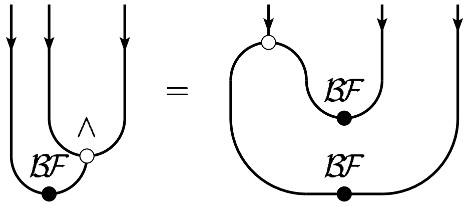
This section comes up with a peculiarity about the Laplace expansion. Rota and Stein introduced a so called Laplace Hopf algebra, which is an augmented connected Hopf algebra where the product may be deformed by cliffordization and the bilinear form permits Laplace expansion [119]. In fact they gave two more relations which we will not consider here. Also Brouder used Laplace Hopf algebras in his work on renormalization [22] which is the main interest of our study too.

We made in this work good use of product co-product duality, and Laplace expansions. This motivated to examine in which way the condition that the wedge product and the Graßmann co-product are related. In form of a tangles product co-product duality for tow given such structures imposes restrictions on $\mathcal{BF}$. This reads in tangles



(7-31)

and



(7-32)

If we write this in algebraic terms using Sweedler notation, we obtain for arbitrary elements $u, v, w \in \bigwedge V$

$$\mathcal{BF}(u \wedge v, w) = \mathcal{BF}(u, w_{(2)})\mathcal{BF}(v, w_{(1)}) \qquad \text{and}$$
$$\mathcal{BF}(u, v \wedge w) = \mathcal{BF}(u_{(1)}, w)\mathcal{BF}(u_{(2)}, v). \tag{7-33}$$

This is in fact the Laplace expansion into rows or columns, where the Hopf gebraic method allows to expand in a single step into a couple of rows or columns. It was shown in [66] that the Laplace pairing implies an exponential extension of B to $B^\wedge$. This was used also in our consideration about associativity. It is obvious from the exponential representation, that bilinear forms can be added in the following way

$$B = g + F$$
$$B^\wedge = e^B \; = \; e^{g+F} \; = \; e^g \, e^F \, e^{\frac{1}{2}[g,F]} \times \ldots \tag{7-34}$$

which is a Hausdorff like formula. Pairings which are obtained from co-boundaries, as will be done below, result in exponentially generated bilinear forms. However, since they have to be calculated via Hausdorff like formulas, also here the Hopf gebra approach is indispensable to manage the complexity of the calculations. On the other hand, it will be interesting to study Hausdorff like formulas on their own right using Hopf gebraic methods. We have to add, that a Clifford product and its dualized co-product do *not* form a Hopf gebra but only a biconvolution, since no antipode exists. This can be circumvented if one introduces different orderings for vectors and co-vectors [59].

7.3 Renormalization group and Z-pairing

7.3.1 Renormalization group

To be able to formulate renormalized time-ordered products Brouder had to introduce renormalization parameters. This is done by using scalar valued linear forms $Z : \bigwedge V \to \Bbbk$. This introduces 2^n independent renormalization parameters and even infinitely many such parameters if an uncountable or contiguous index set is used. We require that the linear forms Z do form a group under the convolution product

$$Z \star Z' = Z''$$
$$Z \star Z^{-1} = u \; = \; \eta \circ \epsilon. \tag{7-35}$$

We assume that the convolution is w.r.t. a Graßmann Hopf gebra or a symmetric such Hopf gebra, which might be called Weyl Hopf gebra. In this case it is possible to deduce the convolutive unit to be $u = \eta \circ \epsilon$. Such Hopf gebras are augmented and connected.

In such cases one is able to define the inverse Z^{-1} of Z by the well established recursive way. We require that Z is normalized and acts trivially on the generators

$$Z(1) = 1 \qquad\qquad Z(e_i) = 0. \tag{7-36}$$

Furthermore, we use the notion of proper cuts indicated by a prime at the sum over the terms of the co-product

$$\Delta'(x) = \Delta(X) - \mathrm{Id} \otimes x - x \otimes \mathrm{Id} \;=\; \sum{}' x_{(1)} \otimes x_{(2)}. \tag{7-37}$$

We stress once more that this is possible for non-interacting Hopf gebras only. From

$$\begin{aligned} Z \star Z^{-1}(x) &= \eta \circ \epsilon(x) \;=\; x_0 \\ &= Z(x_{(1)}) Z^{-1}(x_{(2)}) \\ &= Z^{-1}(x) + Z(x) + \sum{}' x_{(1)} \otimes x_{(2)} \end{aligned} \tag{7-38}$$

one finds

$$Z^{-1} = (x_0 - Z(x)) - \sum{}' x_{(1)} \otimes x_{(2)} \tag{7-39}$$

for the inverse. Especially one finds

$$Z^{-1}(x) = 1 \qquad\qquad Z^{-1}(x) = 0 \quad \forall x \in V \tag{7-40}$$

showing that $Z^{-1}(x)$ belongs to the class of linear forms which we consider.

These linear forms constitute a group under the convolution, which is called *renormalization group*. We ask under which conditions this group is abelian. That is, we want to have $A \star B = B \star A$ for our linear forms. We compute this convolution as follows:

$$\begin{aligned} A \star B(x) &= A(x_{(1)}) B(x_{(2)}) \\ &= (-1)^{|x_{(1)}||x_{(2)}|} A(x_{(2)}) B(x_{(1)}) \\ &= (-1)^{(|x_{(1)}||x_{(2)}| + |A(x_{(2)})||B(x_{(1)})|)} B(x_{(1)}) A(x_{(2)}) \\ &= (-1)^{(|x_{(1)}||x_{(2)}| + |A(x_{(2)})||B(x_{(1)})|)} B \star A(x) \end{aligned} \tag{7-41}$$

We define $\mathbb{Z}_2$-graded maps as follows

$$C : \bigwedge V^{\pm} \to \bigwedge V^{\pm} \tag{7-42}$$

and one obtains in this case that

$$|x| = |C(x)| \qquad \mod 2 \tag{7-43}$$

holds. If the above discussed maps A and B are $\mathbb{Z}_2$-graded, we find for the prefactor

$$(-1)^{|x_{(1)}||x_{(2)}|+|A(x_{(2)})||B(x_{(1)})|} = (-)^{2|x_{(1)}||x_{(2)}|} = +1 \tag{7-44}$$

and the corresponding convolution product is commutative, hence the renormalization group is abelian. Since our maps Z shall be normalized, we have to choose them to be even, i.e. they are trivial on the odd parts

$$Z : \bigwedge^{2n+1} V \to 0. \tag{7-45}$$

Any linear map can be written as a linear operator Z' followed by the counit

$$Z = \epsilon \circ Z' \qquad\qquad Z' : \bigwedge V \to \bigwedge V \tag{7-46}$$

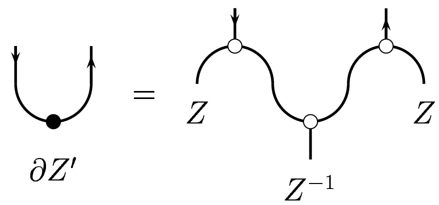
which introduces however additional spurious or 'gauge' parameters due to the projection. Z' can now be written as a bilinear form, that is as a cup tangle by 'bending up' one leg using evaluation, acting then in the following way

$$\partial Z' : \bigwedge V \times \bigvee V^* \to \mathbb{k}, \tag{7-47}$$

i.e. a cup tangle on this 'quantum double' of spaces. This allows to apply a boundary operator which results in the following co-chain, see [91]

$$\partial Z'(u, v^*) = Z(u_{(1)})Z(v^*_{(2)})Z^{-1}(u_{(2)} \wedge v^*_{(1)}). \tag{7-48}$$

In terms of tangles this reads



(7-49)

and one can use commutativity of the convolution product to remove ambiguties at which place, left or right, the linear forms have to be applied. If we introduce a self dual space $\mathbf{V} = V \oplus V^*$, we can neglect the duality, i.e. the arrows in the tangles and arrive at the Z-pairing employed by Brouder.

7.3.2 Renormalized time-ordered products as generalized Clifford products

It was proved in [56] that normal-ordered and time-ordered operator products can be defined as wedge and cliffordized wedge products, where the latter was called dotted wedge product and emerged from a cliffordization w.r.t. a fully antisymmetric bilinear form $F^\wedge$. While the wedge can be identified with the time-ordered products the dotted wedge corresponds to the normal-ordered case. In our previously published works, we had restricted this mechanism to Hamilton formalism, which relays on a one-time formulation [129, 47, 48, 60, 50] for convenience and to make contact to the well established formalism of Stumpf and coll., see [128, 17]. The problem of renormalization is avoided in a different manner by Stumpf, since the main interest was in non-linear spinor field theory, which is non-renormalizable. The transition from multi-time to one-time formulations is intimately connected to renormalization. But, the Hopf gebraic method is purely combinatorial and does not make any assumption about the nature of the coordinates of the fields and applies for multi-time correlation functions too.

In the following we introduce with Brouder [22] some additional pairings which can be used to define cliffordizations and generalized Clifford products. This is done with hindsight to come up with a structure suitable for renormalization. These pairings and the resulting cliffordization are easily established in BIGEBRA and we used this device to check the results given here.

Since we want to have normal- and time-ordered operator products and correlation functions, we have to have a bilinear form $B = g + F$, $g^T = g$, $F^T = -F$ in the case of fermions. As long as algebraic relations are considered, this mechanism works for bosons in a similar manner. In fact the original work of Brouder deals with bosons.

To come up with a generalized product, we have to add to the bilinear form $B^\wedge$ additional parameters, the parameters of renormalization. We use the renormalization group.

Having Z and Z^{-1} established we are able to define the Z-pairing as a product deformation of the time-ordered product, i.e. the wedge. This is the above defined co-chain and reads as follows

$$\begin{aligned} &\partial Z : \bigwedge V \times \bigwedge V \to \Bbbk \circ \mathrm{Id} \\ &\partial Z(x,y) = \sum Z(x_{(1)})Z(y_{(1)})Z^{-1}(x_{(2)} \wedge y_{(2)}). \end{aligned} \tag{7-50}$$

In [48] we had shown that in reordering processes which connect bilinear forms like $g^\wedge$ and $B^\wedge$ no additional divergencies occur! A grade preserving bilinear form $F^\wedge$ which arrises from exponentiating seemed not be able to mediate renormalization. We have

$$\begin{aligned} \mathcal{N}(\psi_1, \ldots, \psi_n) &= \psi_1 \,\dot\wedge\, \ldots \,\dot\wedge\, \psi_n && \text{normal-ordered product} \\ \mathcal{T}(\psi_1, \ldots, \psi_n) &= \psi_1 \wedge \ldots \wedge \psi_n && \text{time-ordered product.} \end{aligned} \tag{7-51}$$

We obtained a ‘transition formula’ from the cliffordization [56]

$$u \,\dot\wedge\, v = F^\wedge(u_{(2)}, v_{(1)})(u_{(1)} \wedge v_{(2)}). \tag{7-52}$$

Which holds in both directions. Choosing the counit of the normal-ordered algebra, i.e. w.r.t. the dotted wedge, to be the vacuum, we note that $\dot\wedge$ is a local product. However already the time-ordered product $\wedge$ is then non-local with great implications for the vacuum structure. Remember that as *algebras* both structures are isomorphic, which probably rendered their distinction to be difficult for a long time. Both such structures are *quantized* due to the introduction of a symmetric bilinear form g extended by exponentiation to $g^\wedge$ in the same manner.

Brouder imposed on the Z-mappings the following restrictions, which we derived above from certain assertions about the properties of the product.

$Z(\mathrm{Id}) = \mathrm{Id}$, $Z(a) = 0 \ \forall a \in V$ and arbitrary else. This results using the recursive formula for the inverse in the following conditions for Z^{-1}. $Z^{-1}(\mathrm{Id}) = \mathrm{Id}$, $Z^{-1}(a) = 0 \ \forall a \in V$. These conditions necessarily establish that the unit Id remains to be the unit of the renormalized time-ordered product algebra and that the anticommutation relations are not altered *on the generators*! Furthermore, Brouder proved, in the case of bosons, that the Z-pairing fulfils the coupling identity which ensures associativity of the product as we have shown above.

Properties of the Z-pairing if used as cup tangle in a cliffordization are:

- Z-cliffordization yields an unital algebra with Id as unit.
- Z-cliffordization preserves associativity, since it fulfils the co-chain or coupling condition.
- The Z-pairing is even, the renormalization group is therefore abelian.
- Z-cliffordization respects the quantization. That is Z-cliffordization does not alter the (anti)commutation relations of the generating space $V \times V$.

Before we can proceed to renormalized time-ordered products, we have to combine the bilinear form obtained from quantization and time-ordering and the Z-pairing. This is done using Hopf gebra methods and yields the total bilinear form $\mathcal{BF}$

$$\mathcal{BF}(u,v) = \partial Z(u_{(1)}, v_{(2)}) B^\wedge(u_{(2)}, v_{(1)}) \tag{7-53}$$

where we arranged the order of the entries in the r.h.s. to avoid crossings. This differs from Brouder, since he used bosonic algebras there were no sign problem for him. This could be called the Hausdorff bilinear form of $B^\wedge$ and ∂Z for reasons given above. The proof that the cliffordization w.r.t. the bilinear form $\mathcal{BF}$ yields the renormalized time-ordered product can be found in [22].

The remaining problem is to fix the multiplicative renormalization constants by some arguments from physics. In fact they have to be chosen to subtract the divergencies emerging in perturbative QFT calculations. A mathematical basis of axioms would however provide a finite theory from the beginning and not order by order. Furthermore, if one is interested in non-Fock vacua, e.g. in QCD for studying confinement etc. or for the calculation of composites, there is no way out of an *a priori* renormalization which should be based on mathematical arguments.

Brouder [22], using results of Pinter [111, 110], has shown that the operator products and correlation function obtained from $\mathcal{BF}$-cliffordization, which includes time-ordering, normal-ordering for the counit, and renormalization mediated by the Z-pairing is equivalent to the Epstein-Glaser renormalization of time-ordered products [46]. This allows us to identify the product $\&r$ with a renormalized time-ordered operator product and the coefficients derived therefrom, see Brouder's paper. The cliffordization results in a tremendous simple formula for the renormalized Green functions. Let $\mathcal{S}$ be the S-matrix, one has

$$\begin{aligned}(\psi_1 \wedge \psi_2 \wedge \mathcal{T}(\mathcal{S}^\wedge(\psi_n)) &= G_{12}\mathcal{T}(\mathcal{S}^\wedge(\psi_{n+2})) \\ (\psi_1 \circ \psi_2 \circ \mathcal{T}^{\text{ren}}(\mathcal{S}^\wedge(\psi_n)) &= G_{12}^{\text{ren}}\mathcal{T}^{\text{ren}}(\mathcal{S}^\wedge(\psi_{n+2})).\end{aligned} \tag{7-54}$$

As a result we see that renormalization is still within the cliffordization scheme but needs a generalized Clifford product based on a bilinear form. This has a tremendous impact e.g. on the structure of higher commutators, the generic antipode, the generic crossing of the renormalized product etc. This structure will be considered in more detail elsewhere.

The Brouder formulation of Epstein-Glaser renormalization of time-ordered products turns out to be *one* but probably not the most general or unique generalization of a Clifford product arising from cliffordization. An axiomatization of renormalized cliffordizations would be most desirable. In this sense our investigations of this chapter are preliminary.

Chapter 8

(Fermionic) quantum field theory and Clifford Hopf gebra

In this chapter we develop a formulation of fermionic quantum field theory (QFT) based on Hopf gebraic methods. We concentrate on fermions, however, the bosonic case runs along the same lines. Bosons will occur in our treatment of spinor quantum electrodynamic. This approach to QFT would have not been possible without the versatile development of functional quantum field theory by Stumpf and coll. [108, 109, 128, 79, 65, 47, 17]. Since the method is readily available in two monographs we will not develop functional QFT here, but deliver only those parts which are necessary for our transition to a Hopf gebraic treatment. It will turn out, that this is not a mere translation. The Hopf gebraic formulation will clearly separate concepts which have not properly been distinguished in conventional QFT. Only this mathematically sound tool allows to deliver powerful and efficient formulas which will be useful in calculations too.

The aim of this chapter is to provide an algebraic skeleton for QFT which is dealing with all peculiarities of QFT as far as the algebraic parts are concerned. Renormalization is treated along the lines of the last chapter following Brouder [22]. However, the present treatment is still not ready-to-use, say for n-th order perturbation calculations in QED. But it is possible already now to formalize a great variety of QF theoretical calculations to such an extend that computer algebra systems as CLIFFORD/BIGEBRA for Maple can be used to evaluate expressions. This includes e.g. the derivation of functional equations or renormalization.

Path integrals are, at this stage of the development, not appropriate, since they are to compact and formal to provide access e.g. to the peculiar combinatorics of renormalization. Nevertheless, since path integrals are formal solutions of the Schwinger-Dyson hierarchy equations it may be possible to come up with a mathematical background for their foundation. Reviewing our techniques, as developed so far in this treatise, it may be conjectured that the path measure will be related to an umbral operator [80]. Such an operator is a linear form on our Hopf gebra having very peculiar combinatorial and algebraic properties. Umbrae cannot be transformed e.g. without taking peculiarities into account etc. Therefore it seems to be convenient to develop firstly the more basic functional differential point of view, but using generating functionals.

8.1 Field equations

To be able to define a QFT we need two informations: (i) a field equation and (ii) (anti)commutation rules. The Lagrangian point of view is more sophisticated and mostly chosen to incorporate symmetries. However it is not essential to the theory. Field equations are formulated in terms of field operators ψ_I, where the multiindex I may contain any sort of indices, including uncountable indices or continuous variables $I = (\Lambda, r, t, \alpha, A, \ldots)$. For a compact presentation, and anticipating the mathematical structure which we are going to use, we include at the same time the index Λ which distinguishes field operators and their conjugated or dual field operators. An analogous notation is chosen for bosonic field operators B_K.

As the name suggests, field operators are elements of an operator algebra. The algebra structure is encoded in the quantization which is given by (anti)commutation relations

$$\{\psi_{I_1}, \psi_{I_2}\}_+ = A_{I_1 I_2} \qquad\qquad [B_{K_1}, B_{K_2}]_- = C_{K_1 K_2}. \tag{8-1}$$

Note that the above relations contain, due to our index doubling, all commutators including the zero ones. It is obvious, that A is symmetric and C is antisymmetric.

An example of a field equation would be a non-linear spinor field, which comes up with the following field equation. This equation is written in a Schrödinger like form which will allow us later to pass easily to generating functionals and functional equations.

$$i\partial_0 \psi_I = D_{I I_1}\psi_{I_1} + g V_I^{\{I_1 I_2 I_2\}_{as}} \psi_{I_1}\psi_{I_2}\psi_{I_3}. \tag{8-2}$$

$D_{I_1 I_2}$ is (almost) the Dirac operator and $V_I^{I_1 I_2 I_3}$ is a local interaction vertex. Obviously this equation is non-renormalizable, but very oftenly used in phenomenological models. A slightly more general spinor equation was used by Stumpf to recover the standard model of elementary particle theory via composite calculations. Functional QFT was developed to manage the problems arising from such a task since it requires non-perturbative methods.

A second example is spinor QED, i.e. a Dirac field coupled to a vector boson, the photon. The field equations are derived from the classical equations

$$\begin{aligned}
&i) \qquad && \partial_\nu F^{\mu\nu} + \frac{ie_0}{2}\Psi C \gamma^\mu \sigma^2 \Psi = 0 \\
&ii) \qquad && (i\gamma^\mu \partial_\mu - m_0)\Psi + e_0 A_\mu \gamma^\mu \sigma^3 \Psi = 0 \\
&iii) \qquad && F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu,
\end{aligned} \tag{8-3}$$

where we have suppressed the indices. C is the charge conjugation matrix of Dirac theory and σ encodes the double index information, taking care if a field is a adjoint or not. Details may be found in [47, 60, 50]. Introducing Coulomb gauge, eliminating the longitudinal part of the vector potential A and introducing an index doubled field B_K for the canonical pair (A_K, E_K) of bosonic fields allows us to write the field equations as

$$\begin{aligned}
i\partial_0 \Psi_{I_1} &= D_{I_1 I_2}\Psi_{I_2} + W^k_{I_1 I_2} B_k \Psi_{I_2} + U_{I_1}^{I_2 I_3 I_4}\Psi_{I_2}\Psi_{I_3}\Psi_{I_4} \\
i\partial_0 B_{K_1} &= L_{K_1 K_2} B_{K_2} + J_{K_1}^{I_1 I_2}\Psi_{I_1}\Psi_{I_2}
\end{aligned} \tag{8-4}$$

where we have used the following abbreviations

$$
\begin{aligned}
I &:= \{\alpha, \Lambda, \mathbf{r}\} \qquad K := \{k, \eta, \mathbf{z}\} \qquad P^{tr} := 1 - \Delta^{-1}\nabla \otimes \nabla \\
D_{I_1 I_2} &:= -(i\gamma_0\gamma^k\partial_k - \gamma_0 m)_{\alpha_1\alpha_2}\delta_{\Lambda_1\Lambda_2}\delta(\mathbf{r}_1 - \mathbf{r}_2) \\
W^K_{I_1 I_2} &:= e_0(\gamma_0\gamma^k)_{\alpha_1\alpha_2}\delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_1 - \mathbf{z})\delta_{1\eta}\sigma^3_{\Lambda_1\Lambda_2} \\
U^{I_2 I_3 I_4}_{I_1} &:= -\frac{i}{8\pi}e_0^2\left[(C\gamma_0)_{\alpha_2\alpha_3}\delta_{\alpha_1\alpha_4}\sigma^2_{\Lambda_2\Lambda_3}\sigma^2_{\Lambda_1\Lambda_4}\frac{\delta(\mathbf{r}_2 - \mathbf{r}_3)\delta(\mathbf{r}_1 - \mathbf{r}_4)}{|\mathbf{r}_1 - \mathbf{r}_2|}\right]_{as\{I_2 I_3 I_4\}} \\
L_{K_1 K_2} &:= i\delta(\mathbf{z}_2 - \mathbf{z}_1)\delta_{k_1 k_2}\delta_{\eta_1 1}\delta_{\eta_2 2} + i\Delta(\mathbf{z}_1)\delta(\mathbf{z}_1 - \mathbf{z}_2)\delta_{k_1 k_2}\delta_{\eta_1 2}\delta_{\eta_2 1} \\
J^{I_1 I_2}_K &:= -\frac{1}{2}e_0 P^{tr}(\mathbf{z} - \mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)(C\gamma^k)_{\alpha_1\alpha_2}\delta_{2\eta}\sigma^2_{\Lambda_1\Lambda_2}
\end{aligned}
\tag{8-5}
$$

and impose canonical quantization

$$
\begin{aligned}
&i) \qquad && \{\Psi_{I_1}, \Psi_{I_2}\}^t_+ := A_{I_1 I_2} = C\gamma_0\sigma^1\delta(\mathbf{r}_1 - \mathbf{r}_2) \\
&ii) && [B_K, \Psi_I]^t_- := 0 \\
&iii) && [B_{K_1}, B_{K_2}]^t_- =: C_{K_1 K_2}.
\end{aligned}
\tag{8-6}
$$

In the above field equation we find the Laplacian $L_{K_1K_2}$, a non-linear self interaction term for the spinor field $U^{I_2I_3I_4}_{I_1}$ which emerges from the longitudinal vector potential, i.e. the Coulomb interaction and *two* coupling terms. One is a boson-fermion coupling $W^K_{I_1I_2}$ and the other is a fermion-boson coupling $J^{I_1I_2}_{K_1}$. Our presentation is not covariant. It is well known that consistency implies that quantization has to be done in Hamilton formulation, which is also not covariant. However, we do not loose any information as long as we perform exact manipulations with this system of equations since we could transform back to the covariant picture. However, in the chosen form it is much easier to distinguish the various terms and to appreciate their physical content.

The task is now to provide a Schwinger-Dyson hierarchy for these field equations from which in principle results can be computed. To do this, one has to pass from a single field to a hierarchy of field equations which will be encoded by a functional.

8.2 Functionals

We will define fermionic functionals in terms of quantum Clifford algebras. The bosonic case runs along the same lines by analogy, as far as no topological assertions have to be made.

A *generating function* is used to encode a sequence of numbers $(a_1, a_2, a_3, \ldots)$ by a function, such that they appear as the coefficients in a polynomial expansion.

$$
\begin{aligned}
f(t) &= \sum \frac{a_n}{n!}t^n \\
\frac{t}{e^t - 1} &= \sum \frac{B_n}{n!}t^n \\
B_n &= \frac{\partial^n}{\partial t^n}\frac{t}{e^t - 1}|_{t=0}
\end{aligned}
\tag{8-7}
$$

where the B_n are the Bernoulli numbers. To get a certain coefficient, we can use the nth-derivative and evaluate it at the point ($t = 0$). The idea is to generalize this technique in that way, that the coefficients are functions, or even distributions.

Having an operator algebra, we need a semi ordering as we have introduced for bases in Clifford algebras in chapter 2. Such ordered monomials span linearly the space on which the operator algebra is built over. If we choose a (semi) ordering $\mathcal{P}$, we obtain the reduced words

$$e_{I_1,\ldots,I_n} := \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}), \tag{8-8}$$

which constitute a basis. Note that such a basis is usually assumed to have the symmetry of the fields, i.e. in our case antisymmetric for fermions and symmetric for bosons. This is not necessary, as we saw in the case of a Clifford basis versus a Graßmann basis, but solely used in QFT. However, see the $U(2)$-model discussed below. The generators of this non-commutative polynomial ring are the *Schwinger sources* j_I for fermions and b_K for bosons. They have to reflect the (anti)symmetry of the ordered fields, since we are interested in Graßmann or Weyl bases. Therefore they have to span Graßmann or Weyl algebras. The derivation operators are written as ∂_I for fermions and δ_K for bosons. The commutation relations are ($\delta_{I_1 I_2}$, $\delta_{K_1 K_2}$ are Kronecker symbols)

$$\{j_{I_1}, j_{I_2}\}_+ \;=\; 0 \;=\; \{\partial_{I_1}, \partial_{I_2}\}_+ \qquad\qquad \{j_{I_1}, \partial_{I_2}\}_+ \;=\; \delta_{I_1 I_2} \tag{8-9}$$

for the fermions, while the bosonic sources fulfil

$$[b_{K_1}, b_{K_2}]_- \;=\; 0 \;=\; [\delta_{K_1}, \delta_{K_2}]_- \qquad\qquad [\delta_{K_1}, b_{K_2}]_- \;=\; \delta_{K_1 K_2}. \tag{8-10}$$

We have adopted the convention to incorporate the factor 2 into the definition of the bilinear forms. This will lead to a factor $1/2$ in contractions, but follows the QFT standard. It is clear, that the j_I sources generate a Graßmann algebra, while the b_K sources do the same for a Weyl algebra or symmetric algebra. We denote by V the space generated by the j_Is and by $\bigwedge V$ the whole space, similar we use W and $\mathrm{Sym}(W)$ for the Weyl algebra. The product between the sources is defined to be the wedge product for the j_Is, the vee-product for the duals ∂_I and juxtaposition for the bosons. The last setting is slightly to narrow, but we will deal with bosons only occasionally.

From the previously obtained results, we know, that these algebras are Hopf gebras. We can even adjoin a unit map η and a counit map ϵ and know that these algebras are bi-augmented bi-connected Hopf gebras. The crossing is the graded switch τ for fermions and the non-graded switch sw for bosons.

The physicist's notation for the counit uses functional 'vacuum' states. One defines

$$\begin{aligned} j_I \mid 0\rangle_F &= 0 \qquad\qquad {}_F\langle 0 \mid \partial_I \\ \epsilon^\wedge(U) &= {}_F\langle 0 \mid \sum_{\text{grades}} U^{I_1,\ldots,I_n} j_{I_1} \wedge \ldots \wedge j_{I_n} \mid 0\rangle_F \\ &= U^0. \end{aligned} \tag{8-11}$$

In fact, $\epsilon^\wedge$ is the projection onto the coefficient of the identity element Id, since usually one assumes normalization so that ${}_F\langle 0 \mid \mathrm{Id} \mid 0\rangle_F = 1$.

To expand a reduced word $e_{I_1,\ldots,I_n}$ into the j-sources, we need a mapping $\mathsf{p} : \bigwedge V \to \Bbbk$ which establishes the ordering $\mathcal{P}$ of the field operators. Before we can proceed, we have to give the definition of the field operators in terms of the sources j, ∂.

We will make use of the Chevalley deformation, knowing that this can be generalized by Hopf gebraic means, and define the field operator to be a Clifford map. Let $e_{I_1,\ldots,I_n}$ be a reduced monomial w.r.t. the ordering $\mathcal{P}$. We have to define two field operators which allow to add a single field operator to this basis monom, one adds from the left and one adds from the right. These field operators read

$$
\begin{aligned}
\psi_I &= \partial_i + \frac{1}{2} P_{IL}\, j_L \wedge \\
\psi_I^{op} &= \partial_i - \frac{1}{2} P_{LI}\, j_L\ \wedge . \qquad \text{(8-12)}
\end{aligned}
$$

Note that the indices of the field operator acting by opposite multiplication are reversed, due to the fact that $\mathcal{C}\ell^{op}(V, P)$ and $\mathcal{C}\ell(V, -P^T)$ are isomorphic. This could be called a Pieri formula for fermionic quantum field theory. The bosonic case is treated similarly. With our pre-knowledge, we identify the action of such an operator as a Clifford product and write a circle for this product, where we leave the ^{op} with the field operator abusing its meaning as opposite product. We obtain

$$
\begin{aligned}
\psi_I \circ e_{I_1,\ldots,I_n} &= e_{I,I_1,\ldots,I_n} \ =\ \mathcal{P}(\psi_I, \psi_{I_1}, \ldots, \psi_{I_n}) \\
\psi_I^{op} \circ e_{I_1,\ldots,I_n} &= e_{I_1,\ldots,I_n,I} \ =\ \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}, \psi_I). \qquad \text{(8-13)}
\end{aligned}
$$

Now, given an element $U \in \bigwedge V$, we can expand it into our j-basis using the projection π_p^a as follows

$$
\begin{aligned}
\pi_p^a(e_{I_1,\ldots,I_n}) &= \pi_p^a\big(\mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n})\big) \\
&= \langle \partial_{I_n} \vee \ldots \vee \partial_{I_1} \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n})\, \rangle_{\pi_p^a}\, j_{I_1} \wedge \ldots \wedge j_{I_n} \mid 0\rangle_F \\
&= \langle 0 \mid \partial_{I_n} \vee \ldots \vee \partial_{I_1} \mathcal{P}(\psi_{I_1}, \ldots, \psi_{I_n}) \mid a\rangle\, j_{I_1} \wedge \ldots \wedge j_{I_n} \mid 0\rangle_F \\
&= \rho(I_1, \ldots, I_n \mid a)\, j_{I_1} \wedge \ldots \wedge j_{I_n} \mid 0\rangle_F . \qquad \text{(8-14)}
\end{aligned}
$$

π_p^a is a sort of grade projection operator. We are able to learn several things from this calculation:

i) The functional vacuum is related to the counit of the Graßmann algebra of the Schwinger sources.

ii) The counit of the Schwinger sources is *not* directly related to the physical vacuum, which we have denoted by a bra-ket notation also. Moreover, this is a second and independent linear form $\langle \ldots \rangle_{\pi_p^a} : \bigwedge V \to \Bbbk$. This linear form may be parametrized by a physical state $|a\rangle$. If this state is the vacuum state, we deal with vacuum expectation values.

iii) The physical vacuum depends also on the ordering $\mathcal{P}$. The mapping $\mathsf{p} : \bigwedge V \to \Bbbk$ is identical to the *correlation functions* $\rho(I_1, \ldots, I_n \mid a)$.

iv) This projection encodes all of the combinatorics of QFT, as we will see below.

The immediate question is, if there is a second Graßmann exterior or Clifford product, which turns the above projection π_p^a into the counit of this algebra. Moreover we have to ask, what kind of product is induced by this type of projection inside the reduced words. In the case of quantum Clifford algebras we know already that we can choose e.g. a Clifford basis, a Graßmann wedge basis or a dotted Graßmann wedge basis.

Note that this projection is exactly the same as we have introduced for the renormalization parameters in the previous chapter. In fact, we can show the following result. Given a mapping $\mathsf{p} : \bigwedge V \to \Bbbk$ we can define an algebra homomorphism as follows. Let $\mathcal{P}$ be the operator acting on $x \in \bigwedge V$ by convolution in the following way (p^{-1} defined as $\mathsf{p}^{-1} \star \mathsf{p} \,=\, \eta \circ \epsilon$)

$$\mathcal{P}(x) = \mathsf{p}(x_{(1)})x_{(2)} \qquad\qquad \mathcal{P}^{-1}(x) = \mathsf{p}^{-1}(x_{(1)})x_{(2)} \tag{8-15}$$

which is assumed to be commutative, i.e.

$$\begin{aligned} \mathcal{P} &= \mathsf{p} \star \mathrm{Id} \;=\; \mathrm{Id} \star \mathsf{p} \\ \mathcal{P}^{-1} &= \mathsf{p}^{-1} \star \mathrm{Id} \;=\; \mathrm{Id} \star \mathsf{p}^{-1}, \end{aligned} \tag{8-16}$$

then the product inside the ordering is obtained by the following homomorphism ($x, y \in \bigwedge V$)

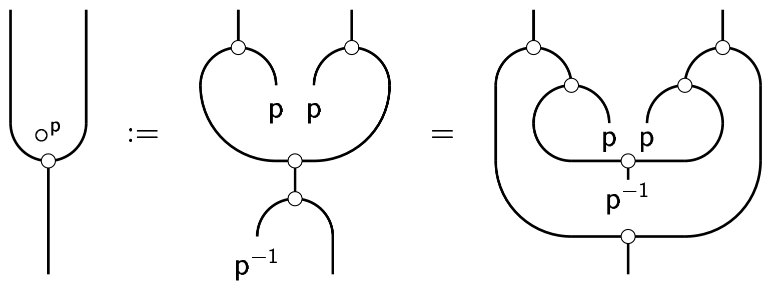
$$\mathcal{P}(x \circ^{\mathsf{p}} y) = \mathcal{P}(x) \wedge \mathcal{P}(y), \qquad \mathcal{P}^{-1}(x \wedge y) = \mathcal{P}^{-1}(x) \circ^{\mathsf{p}} \mathcal{P}^{-1}(y), \tag{8-17}$$

moreover the circle product $\circ^p$ is the cliffordization of the undeformed wedge product w.r.t. the bilinear form (up to a sign)

$$\begin{aligned} \partial\mathcal{P}(u, v) &= \mathsf{p}(u_{(1)})\mathsf{p}(v_{(2)})\mathsf{p}^{-1}(u_{(2)} \wedge v_{(1)}) \\ \partial\mathcal{P}^{-1}(u, v) &= \mathsf{p}^{-1}(u_{(1)})\mathsf{p}^{-1}(v_{(2)})\mathsf{p}(u_{(2)} \wedge v_{(1)}) \end{aligned} \tag{8-18}$$

This bilinear form is a co-chain, as we have discussed in the previous chapter. Indeed, it looks as if the ‘renormalization group’ as discussed there is much more an ‘ordering group’. We will see, that renormalization might be addressed as a sort of residual re-ordering if normal-ordering is done w.r.t. the free propagator. Reorderings are invertible since we demanded that the endomorphisms p do form a group under convolution.

To prove the above given statement, one has to use the fact that $\mathcal{P}(x) = \mathsf{p}(x_{(1)})x_{(2)} = x_{(1)}\mathsf{p}(x_{(2)})$ and that the co-product is an algebra homomorphism and the cocommutativity. In terms of tangles this reads (up to signs):



(8-19)

Hence we can state, under the above given assumptions, that if an algebra morphism is constructed from a linear form by convolution then it induces a cliffordization where the bilinear form is a cochain.

After this preliminary consideration we can write down a generating functional. For this purpose, we add up the reduced monoms including the prefactor, that is the n-point correlation function, to build up a general element of $\bigwedge V$. This reads, if we indicate also the state $|a>$ w.r.t. the transition matrix elements, as

$$\begin{aligned} |\mathcal{P}(j,a)\rangle_F &= \sum_{i=0}^{n} \frac{i^n}{n!} \rho_n(I_1,\ldots,I_n \mid a)\; j_{I_1} \wedge \ldots \wedge j_{I_n} |0\rangle_F \\ \rho_n(I_1,\ldots,I_n \mid a) &=< 0 \mid \mathcal{P}(\psi_{I_1},\ldots,\psi_{I_n})|a> . \end{aligned} \tag{8-20}$$

With respect to the endomorphisms of $\bigwedge V$, this is a 'state' and has thus also transformation properties. The implementation of the Poincaré group symmetries on functional spaces e.g. is discussed in great detail in [128].

8.3 Functional equations

We have now generating functionals at our disposal. The next step is to implement the dynamics on such functionals. They code directly the Schwinger-Dyson hierarchy, which is the hierarchy of the coupled n-point correlation functions. Our goal is to derive a Schrödinger like equation for such functional states. It should however be noted that our basis elements, the reduced words $e_{I_1,\ldots,I_n}$ are neither normalized nor orthogonal and cannot directly be interpreted in physical terms. If one assumes a Fock representation, the ordinary perturbative treatment does apply.

Our starting point is the Heisenberg equation

$$\begin{aligned} i\dot{\psi}_I &= [\psi, H]_- \\ H &= H[\psi]. \end{aligned} \tag{8-21}$$

where $H[\psi]$ is assumed to generate a one parameter family of automorphisms by integrating the above equation

$$\psi_I(t) = e^{iHt}\psi_I(0)e^{-iHt}. \tag{8-22}$$

This equation translates into the following functional equation

$$i\partial_0|\mathcal{P}(j,p)\rangle_F = H[j,\partial]^{\mathcal{P}}|\mathcal{P}(j,p)\rangle_F \tag{8-23}$$

and our task is to calculate the *functional Hamiltonian* $H[j,\partial]^{\mathcal{P}}$, which depends on the chosen ordering. We can use the above defined Clifford maps to perform this task. This results in

$$\begin{aligned} i\dot{e}_{I_1,\ldots,I_n} &= [e_{I_1,\ldots,I_n}, H[\psi]]_- \;=\; e_{I_1,\ldots,I_n}H[\psi] - H[\psi]e_{I_1,\ldots,I_n} \\ &= H[\psi^{op}]e_{I_1,\ldots,I_n} - H[\psi]e_{I_1,\ldots,I_n}. \end{aligned} \tag{8-24}$$

The opposite product allows to write the functional Hamiltonian as an endomorphism acting from the left alone. Having an equation for a reduced word, i.e. a basis monom, we can add up the hierarchy and obtain the functional equation

$$\begin{aligned} i\partial_0 \mid \mathcal{P}(j,a)\rangle_F &= (H[\psi^{op}] - H[\psi]) \mid \mathcal{P}(j,a)\rangle_F \\ &= H[j,\partial]^{\mathcal{P}} \mid \mathcal{P}(j,a)\rangle_F. \end{aligned} \tag{8-25}$$

Since we can read of the functional form of $H[\psi]$ directly from the equations of motion, we are immediately ready to calculate the functional Hamiltonian by replacing the field operators with the appropriate Clifford map. This is done as follows

$$H[j,\partial]^{\mathcal{P}} = H[\psi^{op}] - H[\psi] \;=\; H[\partial - 1/2\, P^T j] - H[\partial + 1/2\, Pj]. \tag{8-26}$$

Since our field equations are in general polynomial in the interaction terms, the calculations can be performed very quickly. In fact, this could be recast in Hopf gebraic form. For the interaction term of the spinor field theory this reads

$$\mathcal{P}^{-1}(V_I^{I_1I_2I_3}\,\psi_{I_1} \wedge \psi_{I_1} \wedge \psi_{I_1}) = V_I^{I_1I_2I_3}\,\mathcal{P}^{-1}(\psi_{I_1}) \,\circ^{\mathsf{P}^{-1}}\, \mathcal{P}^{-1}(\psi_{I_2}) \,\circ^{\mathsf{P}^{-1}}\, \mathcal{P}^{-1}(\psi_{I_3}). \tag{8-27}$$

If we demand, in difference to the requirements for renormalization discussed in the previous chapter, that $\mathcal{P}^{-1}(\psi_{I_1}) = \psi_{I_1}$ we end up with the same term, but w.r.t. the new circle product. The opposite field operators imply a reversion of the circle products, i.e. a right action.

The crucial point in this consideration is, that we assume that the original classical field equations have to be formulated with the same wedge product which we used for forming the generating functionals. In other words, the ordering is an algebra homomorphism which induces in our case quantization and ordering in a single step. The quantization stems for fermions from the symmetric parts of P while the ordering depends on the antisymmetric part. In the case of bosons these symmetries are interchanged. The antisymmetric part will show up in the next section to be related to the propagator of the theory.

8.4 Vertex renormalization

Since we deal with non-linear terms like the above discussed ψ^3 term, one has to ask if there occur ordering problems. Indeed, it is well known from standard treatments, that one has to ‘remove’ vertex singularities. However, such singularities emerge also in the re-ordering from, say time- to normal-ordering and vice versa. It is hence only possible to remove such singular contributions in *one* ordering, say the normal-ordering. We have shown in Ref. [47, 48], that in the present formalism *no additional singularities* emerge from a re-ordering.

The point is, that in the standard treatment one does not write down the product. There, the transition is done simply by adding contraction terms, some of them are diagonal $P_{I_1I_1} \cong P(x,x)$ and diverge. This fact is usually discussed verbally and as a solution one comes up with a vertex renormalization denoted by colons

$$:\; \psi_{I_1}\psi_{I_2}\psi_{I_3} \;:\; = \; \psi_{I_1}\psi_{I_2}\psi_{I_3} + \text{contraction terms} - \text{singularities}. \tag{8-28}$$

Nevertheless, this moves around the singularity only from one picture into another and does not yield a solution of the problem, since if the singularities are substracted in the time-ordered picture, they reappear e.g. in the normal-ordered formulation.

Introducing the proper products allows to get rid of these singularities in *all* orderings. That is, a transition from one picture into another does not introduce new spurious singularities. We calculate

$$\psi_{I_1} \circ^{\mathsf{P}^{-1}} \psi_{I_2} \circ^{\mathsf{P}^{-1}} \psi_{I_3} = \psi_{I_1} \wedge \psi_{I_2} \wedge \psi_{I_3} - P_{I_1 I_2}\psi_{i_3} - P_{I_2 I_3}\psi_{i_1} - P_{I_3 I_1}\psi_{i_2} \tag{8-29}$$

and no diagonal singular terms occur. This outcome motivates to study if this mechanism applies directly to functional equations.

8.5 Time- and normal-ordering

In this section we report our findings from Refs. [108, 47, 48, 60, 50], therefore we do not provide the calculational details, but try to exhibit the newly established Hopf gebraic aspects. The former calculations are already considerable more efficient than the derivation of functional equations by means of e.g. Hausdorff formulas. Our 'replacement' formalism is tied to the exponential representation of the functionals and goes back to Anderson [8][1].

Before we go into the details of examples, we examine the transition from time- to normal-ordered functionals. The time-ordered functional is defined w.r.t. the wedge product and has τ-functions as coefficients. We abbreviate the indices by numbers for convenience.

$$|\mathcal{T}(j,a)\rangle^{\wedge} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \tau_n(1,\ldots,n \mid a) j_1 \wedge \ldots \wedge j_n \mid 0\rangle_F. \tag{8-30}$$

The normal-ordered functional is also *expanded w.r.t. the wedge product* and has the ϕ-functions as coefficients.

$$|\mathcal{N}(j,a)\rangle^{\wedge} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \phi_n(1,\ldots,n \mid a) j_1 \wedge \ldots \wedge j_n \mid 0\rangle_F\,. \tag{8-31}$$

Since both fucntionals are expanded in the *same basis*, it is clear that they are different in their content. In fact, the normal-ordered functional corresponds to one-particle irreducible correlation functions in perturbative QFT. Time- and normal ordered functionals can be related by the expenentiated propagator $F_{I_1 I_2}$ in the following manner

$$|\mathcal{T}(j,a)\rangle^{\wedge} \;=\; e^{-\frac{1}{2}F_{I_1I_2}j_{I_1}\wedge j_{I_2}}\,|\mathcal{N}(j,a)\rangle^{\wedge} \qquad |\mathcal{N}(j,a)\rangle^{\wedge} \;=\; e^{\frac{1}{2}F_{I_1I_2}j_{I_1}\wedge j_{I_2}}\,|\mathcal{T}(j,a)\rangle^{\wedge} \tag{8-32}$$

where we have indicated that the functionals are expanded in the wedge basis. In Hopf gebraic terms, this transiton is mediated by an algebra homomorphism in the following way. Let $\mathcal{N}$ be

[1] I thank C. Brouder for pointing out this reference to me.

the algebra morphism which transformes from time- to normal-ordering by changing the basis from the wedge to the dotted wedge basis

$$\mathcal{N}(u \wedge v) = \mathcal{N}(u) \,\dot{\wedge}\, \mathcal{N}(v). \tag{8-33}$$

Applying this operation to a time ordered functional yields the normal-ordered such functional

$$\mathcal{N}\big(\,|\mathcal{T}(j,a)\rangle^{\wedge}\,\big) = |\mathcal{N}(j,a)\rangle^{\dot{\wedge}}\,. \tag{8-34}$$

But this functional has the *same* expansion coefficients, i.e. it stays with the τ-functions! Only after we have re-expanded the normal-ordered functional into the wegde basis, we end up with the above given result. This reads explicitely as

$$\begin{aligned} \mid \mathcal{N}(j,a)\rangle^{\dot{\wedge}} &= \sum \frac{i^n}{n!}\,\tau_n(1,\ldots,n \mid a)\; j_{I_1} \,\dot{\wedge}\, \ldots \,\dot{\wedge}\, j_{I_n} \mid 0\rangle_F . \\ \mid \mathcal{N}(j,a)\rangle^{\wedge} &= \sum \frac{i^n}{n!}\,\phi_n(1,\ldots,n \mid a)\; j_{I_1} \wedge \ldots \wedge j_{I_n} \mid 0\rangle_F\,. \end{aligned} \tag{8-35}$$

The two expansions shows, that the connenction between the τ-functions and the ϕ-functions is not directly mediated by the reordering, but by the re-expansion of the re-ordered functionals in the wedge basis. This is not an artifact of the theory, but is closely related to the fact the we have to chose a unique counit which acts as a projection onto the identity element. This counit depends on the chosen basis. If we use $\epsilon^{\wedge}$ we have to expand *all* functionals in this particular wedge basis. This does on the other hand imply, that we have to use different ordering mappings (linear forms on $\bigwedge V$) e.g. $\mathtt{t} : \bigwedge V \to \Bbbk$ and $\mathtt{n} : \bigwedge V \to \Bbbk$ for obtaining the time- or normal-ordered correlation functions. In this functions one encodes thereby the information about the physical vaccum structure of the theory. This should not be confused with the functional 'vacuum' $|0\rangle_F$ which does not contain physical informations.

8.5.1 Spinor field theory

We start considering the non-linear spinor field theory. Its Hamiltonian is displayed as

$$H[\psi]^{\wedge} = \frac{1}{2} A_{I_1 I_3} D_{I_3 I_2} \psi_{I_1} \wedge \psi_{I_2} + \frac{g}{4} A_{I_1 I_5} V_{I_1}^{I_2 I_3 I_4} \psi_{I_1} \wedge \psi_{I_2} \wedge \psi_{I_3} \wedge \psi_{I_4} \tag{8-36}$$

where we have introduced explicitly the wedge product. Remember that $A_{I_1 I_2}$ is the anticommutator of the fields in index doubled formulation. Since we know, that a Clifford map $\partial \pm 1/2\, Aj\wedge$ is a map into the Clifford algebra $\mathcal{C}\ell(V,A)$ and $\mathcal{C}\ell^{op}(V,A) = \mathcal{C}\ell(V,-A^T)$, we are able to identify the corresponding cliffordization with the time-ordered functional equation. The field operators have to be chosen as

$$\begin{aligned} \psi &= \frac{1}{i}\partial_i - \frac{i}{2} A_{II_1} j_{I_1} \wedge \\ \psi^{op} &= \frac{1}{i}\partial_i + \frac{i}{2} A_{II_1} j_{I_1} \wedge \end{aligned} \tag{8-37}$$

to stay compatible with QF theoretic conventions. Note that this differs from our previously chosen conventions by a swap between product and opposite product resulting just in a renaming of the fields. If we assume the state $|a\rangle$ to be an eigenstate of the Hamiltonian, we end up with the renormalized energy eigen-functional equation

$$\begin{aligned} E_{a0}|\mathcal{T}(j,a)\rangle_F^\wedge = \Big\{ & D_{I_1I_2} j_{I_1} \partial_{I_2} \\ & + g j_{I_1} V_{I_1}^{I_2I_3I_4} \Big(\partial_{I_3}\partial_{I_2}\partial_{I_1} + \frac{1}{4} A_{I_4I'_4} A_{I_3I'_3} j_{I'_4} j_{I'_3} \partial_{I_2} \Big) \Big\} |\mathcal{T}(j,a)\rangle_F^\wedge \end{aligned} \tag{8-38}$$

where the energy value $E_{a0} = E_a - E_0$ is the difference of the energy of the state $|a\rangle$ w.r.t. the vacuum energy E_0 and thus renormalized. One identifies the terms as follows: The D term is the kinetic part of the Dirac operator, the V term has two parts, the interaction part $j\partial^3$ and a quantization part $j^2\partial^2$. This functional equation is time-ordered.

However, for composite calculation one needs the normal-ordered functional equation. Usually this is obtained by the deviation over the intermediate step of the time-ordered equation. But we have another opportunity, we can simply introduce a different cliffordization based on another ordering. Two orderings which are based on antisymmetric operator products or correlation functions can differ only by an antisymmetric part. We can use the *propagator* of the theory to perform this transition. The field operators translate as

$$\begin{aligned} \psi^{op} &= \frac{1}{i}\partial + \frac{i}{2} A j \,\dot{\wedge}\, = \frac{1}{i}\partial + \frac{i}{2} A j \wedge + i F j \wedge \\ \psi &= \frac{1}{i}\partial - \frac{i}{2} A j \,\dot{\wedge}\, = \frac{1}{i}\partial - \frac{i}{2} A j \wedge + i F j \wedge \end{aligned} \tag{8-39}$$

and we can easily calculate the normal-ordered energy functional equation in a *single step* as

$$\begin{aligned} E_{a0}|\mathcal{N}(j,a)\rangle_F^\wedge &= H[j,\partial] \,\dot{\wedge}\, |\mathcal{N}(j,a)\rangle_F^\wedge \\ &= \Big\{ D_{I_1I_2} j_{I_1}\partial_{I_2} - D_{I_1I_3} F_{I_3I_2} j_{I_1} j_{I_2} \\ &+ g V_{I_1}^{I_2I_3I_4} \Big[j_{I'_1} \partial_{I_4}\partial_{I_3}\partial_{I_2} - 3F_{I_2I'_2} j_{I_1} j_{I'_2} \partial_{I_3}\partial_{I_4} \\ &+ (3F_{I_3I'_3} F_{I_2I'_2} + \frac{1}{4} A_{I_3I'_3} A_{I_2I'_2}) j_{I_1} j_{I'_3} j_{I'_2} \partial_{I_4} \\ &+ (3F_{I_3I'_3} F_{I_2I'_2} + \frac{1}{4} A_{I_3I'_3} A_{I_2I'_2}) F_{I_1I'_1} j_{I_1} j_{I'_3} j_{I'_2} j_{I'_1} \Big] |\mathcal{N}(j,a)\rangle_F^\wedge . \end{aligned} \tag{8-40}$$

This equation is of greater complexity, but has to be taken as starting point for e.g. composite calculations. One finds the same terms as in the time-ordered case, but also new terms constituting exchange and quantization terms.

8.5.2 Spinor quantum electrodynamics

Dealing with a coupling theory is slightly more involved. While we can immediately apply our method also to the bosonic field operators B_K, we have to reconsider the commutation relations for bosons and fermions

$$
\begin{aligned}
&i) &&\{\Psi_{I_1}, \Psi_{I_2}\}^t_+ := A_{I_1I_2} = C\gamma_0\sigma^1\delta(\mathbf{r}_1 - \mathbf{r}_2)\\
&ii) &&[B_K, \Psi_I]^t_- := 0\\
&iii) &&[B_{K_1}, B_{K_2}]^t_- =: C_{K_1K_2}.
\end{aligned}
\tag{8-41}
$$

The second equation states that the bosons are considered to be elementary and are not functions of the fermionic fields. If we try to derive the Hamiltonian, we find two coupling terms. It was shown in [129] that a consistency condition is required to ensure that they occur. It is a remarkable fact, that this condition arises from the fact that one has to demand that the functional equations are independent on the ordering of the bosons and fermions vice versa

$$
\begin{aligned}
|\mathcal{P}(a,j,b)\rangle^\wedge = \sum_{n,m=0}^{\infty} \frac{i^n}{n!m!}\rho^s(I_1,\ldots,I_n,K_1,\ldots,K_m \mid a)\times\\
\times\ j_{I_1}\wedge\ldots\wedge j_{I_n}\ b_{K-1}\ldots b_{K_m}\mid 0\rangle_F
\end{aligned}
$$

$$
\begin{aligned}
\rho^1(I_1,\ldots,I_n,K_1,\ldots,K_m \mid a) &= \langle 0 \mid \mathcal{P}^f(\psi_{I_1},\ldots,\psi_{I_n})\mathcal{P}^b(B_{K_1},\ldots,B_{K_m}) \mid a\rangle\\
\rho^2(I_1,\ldots,I_n,K_1,\ldots,K_m \mid a) &= \langle 0 \mid \mathcal{P}^b(B_{K_1},\ldots,B_{K_m})\mathcal{P}^f(\psi_{I_1},\ldots,\psi_{I_n}) \mid a\rangle
\end{aligned}
\tag{8-42}
$$

We denoted the ordering by $\mathcal{P}$ which specialized to $\mathcal{P}^f$ for fermions and $\mathcal{P}^b$ for bosons. The requirement that the functional equations for the ρ^1 hierarchy is equivalent to that of the ρ^2 correlation function results in the following *reaction relation* which ensures that action and reaction between fermions and bosons are mutually equal

$$
C_{K_1K}W^K_{I_1I_2}\Psi_{I_2} = 2A_{I_1I}J^{II_2}_K\Psi_{I_2}.
\tag{8-43}
$$

It is remarkable, that from this equation one is able to *compute* the anticommutator $C_{K_1K_2}$ if the commutator of the fermions $A_{I_1I_2}$ is given and vice versa, see [108, 129]. Finally this relation can be used to eliminate one of the interaction terms in the Hamiltonian in favour of the other, we choose

$$
\begin{aligned}
H[\Psi,B]^\wedge = &\frac{1}{2}A_{I_1I_3}D_{I_3I_2}\Psi_{I_1}\wedge\Psi_{I_2} + \frac{1}{2}A_{I_1I_3}W^K_{I_3I_2}B_K\Psi_{I_1}\wedge\Psi_{I_2}\\
&+\frac{1}{4}A_{I_1I_5}U^{I_2I_3I_4}_{I_5}\Psi_{I_1}\wedge\Psi_{I_2}\wedge\Psi_{I_3}\wedge\Psi_{I_4}\\
&+\frac{1}{2}C_{K_1K_3}L_{K_3K_2}B_{K_1}B_{K_2}.
\end{aligned}
\tag{8-44}
$$

The analogous transition to the functional equation yields for the time-ordered energy equation

$$
\begin{aligned}
E_{0a}|\mathcal{T}(a,j,b)\rangle^\wedge = \big\{&D_{I_1I_2}j_{I_1}\partial_{I_2} + W^K_{I_1I_2}j_{I_1}\partial_{I_2}\partial^b_K + L_{K_1K_2}b_{K_1}\partial^b_{K_2}\\
&+U^{I_2I_3I_4}_{I_1}j_{I_1}(\partial_{I_2}\partial_{I_3}\partial_{I_4} - \frac{1}{4}A_{I_3I_3'}A_{I_2I_2'}j_{I_2'}j_{I_3'}j_{I_4})\\
&+J^{I_1I_2}_K b_K(\partial_{I_1}\partial_{I_2} + \frac{1}{4}A_{I_1I_1'}A_{I_2I_2'}j_{I_1'}j_{I_2'})\big\}|\mathcal{T}(a,j,b)\rangle^\wedge.
\end{aligned}
\tag{8-45}
$$

If we are interested in the normal-ordered energy equation, where only the fermions are normal-ordered, we have to add the propagator term in the Clifford map for the fermionic field operators and get as result

$$\begin{aligned} E_{0a}|\mathcal{N}(a,j,b)\rangle^{\wedge} = \Big\{ & D_{I_1I_2} j_{I_1} \partial_{I_2} - D_{I_1I_3} F_{I_3I_2} j_{I_1} j_{I_2} \\ & + W^K_{I_1I_2} \big[j_{I_1} \partial_{I_2} - F_{I_2I_2'} j_{I_1} j_{I_2'} \big] \partial^b_K \\ & + J^{I_1I_2}_k b_K \big[\partial_{I_1} \partial_{I_2} - 2F_{I_1I_1'} j_{I_1'} \partial_{I_2} \\ & + (F_{I_1I_1'} F_{I_2I_2'} + \frac{1}{4} A_{I_1I_1'} A_{I_2I_2'}) j_{I_1'} j_{I_2'} \big] \\ & + U^{I_1I_2I_3}_J j_J \big[\partial_{I_1} \partial_{I_2} \partial_{I_3} - 3F_{I_3I_4} j_{I_4} \partial_{I_2} \partial_{I_1} \\ & + (3F_{I_3I_4} F_{I_2I_5} + \frac{1}{4} A_{I_3I_4} A_{I_2I_5}) j_{I_4} j_{I_5} \partial_{I_1} \\ & - (F_{I_3I_4} F_{I_2I_5} F_{I_1I_6} + \frac{1}{4} A_{I_3I_4} A_{I_2I_5} A_{I_1I_6}) j_{I_4} j_{I_5} j_{I_6} \big] \\ & + L_{K_1K_2} b_{K_1} \partial^b_{K_2} \Big\} |\mathcal{N}(a,j,b)\rangle^{\wedge} . \end{aligned} \tag{8-46}$$

This equation can be used as a starting point to calculate positronium bound states, see Ref. [60].

As a general rule, we see from this calculations, that one can perform the following clif-fordization process in Hopf gebraic terms

$$\mathcal{P}(H[\psi,B]^{\wedge}) = H[j,\partial,b,\delta]^{\circ_{\mathsf{p}^f}\circ_{\mathsf{p}^b}} \tag{8-47}$$

where p^f and p^b are scalar valued ordering and quantization maps inducing the cliffordization in the fermion and boson sectors. This structure will be investigated elsewhere.

8.5.3 Renormalized time-ordered products

We have discussed already in the previous chapter the method, introduced by Brouder [22], which allows to rewrite Epstein-Glaser renormalization in Hopf algebraic terms. The Epstein-Glaser formalism comes up with a renormalized time-ordered product in position space, while the BPHZ renormalization, also employed by Connes and Kreimer [81, 82, 34, 35], resides in momentum space. It was Pinter who established a clear and to Hopf algebras related formulation of the Epstein-Glaser theory [111, 110]. Finally Brouder realized that this mechanism is a disguised cliffordization. In our formalism, we have to add simply a new bilinear form Z which introduces the renormalization parameters. Since the reorderings, including the renormalization, form a group under convolution, we can introduce an operator $\mathcal{Z}$ and a linear form z as done above to introduce the renormalization. The renormalized time ordered functional is then achieved by

$$\mathcal{Z}\big(|\mathcal{T}(j,a)\rangle^{\wedge} \big) = |\mathcal{Z}(j,a)\rangle^{\circ_{\mathsf{z}}} = |\mathcal{Z}(j,a)\rangle^{\wedge} \tag{8-48}$$

where the last step introduces renormalization in the correlation functions by re-expanding the functional. The whole combinatorics of this process is encoded in this singe and harmless looking equation!

The crucial point is to investigate what kind of physical reason is behind this additional reordering. There are two possibilities:

(i) Since usually one reorders by the free propagator and not w.r.t. the exact propagator, there is a deficit in the ordering process which leads to singularities and has to be removed. In other words, one would expect to obtain no singularities at all if one would use the exact propagator of the theory.

(ii) We have studied generalized cliffordizations in the previous chapter. It might be possible, that renormalization is needed due to the fact, that the quantization and reordering process which is usually performed can come up only with exponentially generated bilinear forms, e.g. we had the extension of B into $B^{\wedge}$. Such bilinear forms are related to theories which possess *only* two-particle interactions. If physics needs *inevitably* non-exponentially generated bilinear forms, such a contribution can be introduced by renormalization and the Z-pairing.

These possibilities will be studied elsewhere.

8.6 On the vacuum structure

While the preceding sections dealt with realistic QF theories, we will discuss the peculiarities occuring from the vacuum structure in a $U(1)$- and $U(2)$-model. This will allow to be very explicite while being not bothered with complications of a realistic theory. But, already the $U(2)$- model, if it is considered as describing a fiber on the space of modes, is a realistic model of BCS superconductivity and provides even generalizations. A detailed exposition including the relation to an analogous C^{*}-algebraic treatment can be found in Ref. [55]. It was in fact this work which initiated the study of time- and normal-ordering and generally QFT in Clifford Hopf gebraic terms.

8.6.1 One particle Fermi oscillator, $U(1)$

In this section we study the simplest possible model, which consists of a single fermionic particle. We are interested in the degrees of freedom of the fiber only, so we suppress a momentum index, which could however be added without altering our consideration. The CAR algebra of a single fermion is created by two generators $\{a, a^{\dagger}\}$ which we denote also by $\{e_1, e_2\}$ in the index doubled formulation, i.e. the index describes the adjointness of the operator. The CAR relations read

$$\{a, a^{\dagger}\}_{+} = \mathrm{Id} \qquad\qquad \text{others zero.} \tag{8-49}$$

The adjoint map is the algebra antihomomorphism which interchanges a and $a^{\dagger}$. This algebra can be turned into a C^{*}-algebra.

Reformulating the CAR relation in Clifford Hopf gebraic terms does not allow to fix the bilinear form B, but only its symmetric part, which encodes thereby the quantization. We introduce therefore a parameter ν, which represents the antisymmetric part, in the following way

$$\begin{aligned}
[B_\nu] = \begin{vmatrix} 0 & \nu \\ 1-\nu & 0 \end{vmatrix} &= \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} + \begin{vmatrix} 0 & -\frac{1}{2}+\nu \\ \frac{1}{2}-\nu & 0 \end{vmatrix} \\
&= [g] + [F_\nu].
\end{aligned} \tag{8-50}$$

This form is chosen for convenience to be able to make contact to C^*-algebraic calculations done by Kerschner [79] and ourselves [54]. The cliffordization is performed w.r.t. this bilinear form. We denote the Clifford product by juxtaposition and the contraction is given as $\rfloor_v$ to indicate its dependence on the parameter. The underlying space of the Graßmann algebra is based on the wedge $\wedge$ product.

It is an easy task to recompute the CAR relations in Clifford terms

$$\begin{aligned}
e_i e_j + e_j e_i &= B_\nu(e_i, e_j) + B_\nu(e_j, e_i) \\
&= 2\, g(e_i, e_j) = \delta_{i,n+1-j}
\end{aligned} \tag{8-51}$$

showing that only the symmetric part of B_ν enters the quantization.

We want to investigate the meaning of the parameter ν. For this purpose, we introduce the counit $\epsilon^\wedge$ and compute the *'vacuum' expectation values* of the elements in a Clifford basis. Remember, that the operator product is given by the Clifford product and that physicists do commonly write down only expressions using this product. We get

$$\begin{aligned}
\epsilon^\wedge(\mathrm{Id}) &= 1 \qquad \text{normalization} \\
\epsilon^\wedge(a\, a^\dagger) &= \epsilon^\wedge(e_1 \circ e_2) = \epsilon^\wedge(e_1 \wedge e_2 + B_{12}) = \nu \\
\epsilon^\wedge(a) &= 0 \\
\epsilon^\wedge(a^\dagger) &= 0.
\end{aligned} \tag{8-52}$$

If we require that our state is positive, we get from the above result and form $\epsilon^\wedge(a^\dagger\, a) = 1-\nu$ the condition $0 \leq \nu \leq 1$ for ν or equivalently $\det(B_\nu) < 0$. The convex set of positive, normalized, linear functionals on the CAR algebra is thus parameterized by $\nu \in [0,1]$.

The reader should note the difference in the description, while the usual treatment comes up with a variety of states acting on a fixed operator algebra, we have a unique state, the *counit* and parameterize the operator product by adding the antisymmetric part F_ν.

In physics, one introduces a Fock vacuum by the following requirement

$$a|0\rangle_{\mathcal{F}} = 0. \tag{8-53}$$

This, however does fix the value ν immediately! One finds $0 = \epsilon^\wedge(a^\dagger\, a) = \epsilon^\wedge(1 - a\, a^\dagger) = 1 - \nu$ and hence $\nu = 1$. A basis of the algebra under this condition is given by the Fock space basis

$$\{\, |0\rangle_{\mathcal{F}}\,,\ a^\dagger\, |0\rangle_{\mathcal{F}}\,\} \tag{8-54}$$

which is two dimensional, and in fact a spinor representation. However, our treatment is totally arbitrary w.r.t. the name of the operators, and we could have introduced a dual Fock space demanding that

$$a^{\dagger}|0\rangle_{\mathcal{F}^*} = 0 \tag{8-55}$$

which would have resulted in the basis

$$\{\,|0\rangle_{\mathcal{F}^*}\,,\; a\,|0\rangle_{\mathcal{F}^*}\,\}\,, \tag{8-56}$$

the span of which we call dual Fock space. It can be shown that this setting corresponds to the parameter $\nu = 0$. What happens for $\nu \in]0,1[$?

While we found two dimensional representations for $\nu = 1$ and $\nu = 0$, we get a 4-dimensional representation in the general case, rendering the algebra to be indecomposable. In other words,

$$a\,a^{\dagger} \quad \text{and} \quad a^{\dagger}\,a \tag{8-57}$$

are almost idempotents if and only if $\nu = 1$ and $\nu = 0$. States with $\nu \in [0,1[$ can be described as linear combinations of this two states and come up to be mixed states. It can be shown, that the time-ordered case is obtained if $\nu = 1/2$, in which case the antisymmetric part F_v of our bilinear form is not present. Renormalization does not make any sense in this almost to trivial example. Since we can come up with a particle number operator which acts on the operators

$$\begin{aligned} [N,a]_- &= -a \\ [N,a^{\dagger}]_- &= \quad a^{\dagger} \end{aligned} \tag{8-58}$$

we call this a $U(1)$-model. It turns out, that N depends on ν in the following way

$$N = (v - \frac{1}{2}) - e_1 \wedge e_2 \;=\; \nu\,\mathrm{Id} + a\,a^{\dagger}. \tag{8-59}$$

This is a Lie group generator only if $\nu = 1/2$, otherwise one deals with a central extended Lie group.

A detailed study of families of idempotents parameterized by a parameter ν will be given elsewhere.

8.6.2 Two particle Fermi oscillator, $U(2)$

While the one particle case is not very interesting, we gain a resonable interesting model already in the next dimension, having two particles and hence four creation and annihilation operators. We have the CAR relations ($\alpha, \beta \in (1,2)$)

$$\begin{aligned} \{a_{\alpha}, a_{\beta}\} &= 0 = \{a^{\dagger}_{\alpha}, a^{\dagger}_{\beta}\} \\ \{a_{\alpha}, a^{\dagger}_{\beta}\} &= \delta_{\alpha,\beta}\mathrm{Id} \end{aligned} \tag{8-60}$$

Table 1.

No:	$A \in CAR$	s_3	$s(s+1)$	q
g_1	Id	0	0	0
g_2	$\frac{1}{2}(a_1a_1^\dagger + a_2a_2^\dagger)$	0	0	0
g_3	$a_1a_2a_2^\dagger a_1^\dagger$	0	0	0
g_4	$a_1a_2^\dagger$	1	2	0
g_5	$\frac{1}{2}(a_1a_1^\dagger - a_2a_2^\dagger)$	0	2	0
g_6	$a_2a_1^\dagger$	-1	2	0
g_7	a_1	$\frac{1}{2}$	$\frac{3}{4}$	1
g_8	$a_1a_2a_2^\dagger$	$\frac{1}{2}$	$\frac{3}{4}$	1
g_9	a_2	$-\frac{1}{2}$	$\frac{3}{4}$	1
g_{10}	$a_2a_1a_1^\dagger$	$-\frac{1}{2}$	$\frac{3}{4}$	1
g_{11}	$a_2^\dagger$	$\frac{1}{2}$	$\frac{3}{4}$	-1
g_{12}	$a_1a_1^\dagger a_2^\dagger$	$\frac{1}{2}$	$\frac{3}{4}$	-1
g_{13}	$a_1^\dagger$	$-\frac{1}{2}$	$\frac{3}{4}$	-1
g_{14}	$a_2a_2^\dagger a_1^\dagger$	$-\frac{1}{2}$	$\frac{3}{4}$	-1
g_{15}	a_1a_2	0	0	2
g_{16}	$a_1^\dagger a_2^\dagger$	0	0	-2

Table 8.1: Eigenvectors of the $U(2)$ and their $U(2)$ quantum numbers. Operator products are Clifford products.

which we will encode once more by index doubling as $\{a_\alpha, a_\beta, a_\alpha^\dagger, a_\beta^\dagger\} = \{e_1, e_2, e_3, e_4\}$. While in the $U(1)$-model we had only a single operator at our disposal, we can implement in the 2-dimensional case a $U(2)$ action. Let N, S_k be the generators of the Lie group $U(2)$, we define

$$\begin{aligned}
&[N, S_k] = 0, \quad [S_k, S_l] = i\epsilon_{klm}S_m \\
&S^\dagger = S, \quad N^\dagger = N \\
&[S_k, a_\alpha] = \sigma_k^{\alpha\beta} a_\beta, \quad [S_k, a_\alpha^\dagger] = \hat{\sigma}_k^{\alpha\beta} a_\beta^\dagger \\
&[N, a_\alpha] = +a_\alpha, \quad [N, a_\alpha^\dagger] = -a_\alpha^\dagger\,.
\end{aligned} \tag{8-61}$$

This are the defining relation of the $U(2)$ generators, two reality conditions and finally their action on the CAR generators. The relations are not independent. A basis using operator products(!) can be given by looking for the eigen states of $S_3, \sum S_k S^k$ and N, which we denote as $(s_3, s(s+1), q)$.

We ask for a linear form $\omega_{\nu w}$ which is positive and normalized. Such a linear form is characterized by its action on a basis and we have to compute the expectation values w.r.t. this form for all 16 states as given in table 8.1. However, we are interested in such states only which are invariant under the $U(2)$ action. Under this requirement we find for the non-zero expectation values

$$
\begin{aligned}
\omega_{\nu w}(\mathrm{Id}) &= 1 \\
\omega_{\nu w}(a_1\, a_1^\dagger) &= \nu \\
\omega_{\nu w}(a_2\, a_2^\dagger) &= \nu \\
\omega_{\nu w}(a_1\, a_2\, a_2^\dagger\, a_1^\dagger) &= w.
\end{aligned}
\tag{8-62}
$$

This is the result of Kerschner [79], which he obtained by C^*-algebraic considerations. However, note that the above basis is not antisymmetric and the operator products cannot be seen as Wick monomials or correlation functions. This fact will lead below to a *renormalization* of the above displayed expectation values.

Let us introduce a bilinear form which allows us to cliffordize the Graßmann algebra over the 4 generators $\{e_i\}$ in such a manner, that the CAR relations hold. The most general form is

$$
[B(e_i, e_j)] = \begin{vmatrix} 0 & u & q & r \\ -u & 0 & s & t \\ -q & 1-s & 0 & m \\ 1-r & -t & -m & 0 \end{vmatrix}. \tag{8-63}
$$

This yields a quantum Clifford algebra and our 'vacuum' state is the counit w.r.t. the wedge product $\epsilon^\wedge$. We want to express the $U(2)$ generators in terms of the generators. From the CAR relations one obtains that

$$
Q' := a_1 a_1^\dagger + a_2 a_2^\dagger. \tag{8-64}
$$

However, this operator has a non-vanishing expectation value and we have to *renormalize* it. This reads

$$
Q = a_1 a_1^\dagger + a_2 a_2^\dagger - (r+s)\,\mathrm{Id} = a_1 \wedge a_1^\dagger + a_2 \wedge a_2^\dagger. \tag{8-65}
$$

From this display we see that the operator Q has to be defined in the wedge basis. The same applies for the basis vectors in our above given table and the other $U(2)$ generators. We find

$$
\begin{aligned}
g_4' &= g_4 - q, & g_5' &= g_5 - \frac{1}{2}(r-s) \\
g_6' &= g_6 - t, & g_{15}' &= g_{15} - u \\
g_{16}' &= g_{16} + m\,.
\end{aligned}
\tag{8-66}
$$

After this renormalization we can *derive* the parameters (ν, w) of the 'vacuum' state $\omega_{\nu w}$ from the data given by the bilinear form that is from quantization and from the 'propagator' which enter the cliffordization process. While the symmetric part is obtained due to canonical quantization,

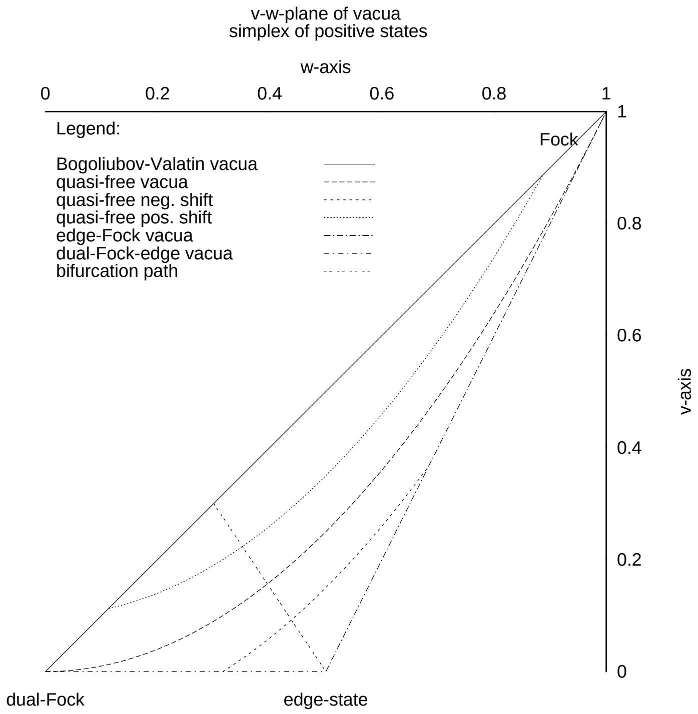


Figure 8.1: Plane of vacuum states for an $U(2)$-model.

the antisymmetric part was in QFT related to the propagator. In our $U(2)$ example, the propagator is given as

$$[F_{ij}] = [\langle \frac{1}{2}[a_i, a_j^\dagger]\rangle_0^\wedge] = \begin{vmatrix} -r+1/2 & q \\ t & -s+1/2 \end{vmatrix}. \tag{8-67}$$

This shows, that ν and w are functions of the parameters r, s, q, t. This model was discussed in great detail, in Ref. [55], including its direct link to BCS theory, the gap equation etc. The reader is invited to consult the original source for this details.

To complete our discussion we want to study the vacuum structure of the present $U(2)$-model a little further. First of all, we consider under which condition the state $\omega_{\nu w}$ is positive. An analogous consideration as above yields

$$\begin{aligned} &0 < w < \nu < 1 \\ &2\nu - 1 < w\,. \end{aligned} \tag{8-68}$$

We draw a diagram, see Figure 8.1, where every point in the affine Euclidean plane corresponds to a state $\omega_{\nu w}$. The positive states form a triangle. We want to discuss the states in and on the borders of the triangle.

Let us impose the Fock and dual Fock space conditions and see which point in the plane corresponds to it

$$\begin{aligned} \prod_{i\in I} a_i \mid 0 >_{\mathcal{F}} =\ & 0 \qquad\qquad \text{Fock space} \\ \prod_{i\in I} a_i^\dagger \mid 0 >_{\mathcal{F}^*} =\ & 0 \qquad\qquad \text{dual Fock space.} \end{aligned} \tag{8-69}$$

We obtain e.g. for the Fock space condition the following expectation values

$$0 =< a_i^\dagger a_i >_{\mathcal{F}} =< \mathrm{Id} - a_i a_i^\dagger >_{\mathcal{F}} =\ 1 - \begin{cases} r & i = 1 \\ s & i = 2\,. \end{cases} \tag{8-70}$$

from which we deduce $\nu = 1$. Furthermore we find

$$\begin{aligned} 0 &=< a_2^\dagger a_1^\dagger a_1 a_2 >_{\mathcal{F}} =< a_2^\dagger a_2 - \mathrm{Id} + a_1^\dagger a_1 + a_1 a_2 a_2^\dagger a_1^\dagger >_{\mathcal{F}} \\ &= 0 - 1 + 0 + w, \end{aligned} \tag{8-71}$$

which yields $w = 1$. An analogous computation can be done for the dual Fock case, and we get

$$\omega_{\mathcal{F}} = \omega_{\nu w}\Big|_{\nu=1\, w=1} =\ \omega_{11}, \qquad\qquad \omega_{\mathcal{F}^*} = \omega_{\nu w}\Big|_{\nu=0\, w=0} =\ \omega_{00}. \tag{8-72}$$

Hence we can identify the up-right and down-left edges of the trinagle of sets to be the Fock and dual Fock state. The representation space is in both cases 4-dimensional and reads as follows

$$\begin{aligned} \mathcal{V}_{\mathcal{F}} &= \{\mathrm{Id} \mid 0 >_{\mathcal{F}}, a_1^\dagger \mid 0 >_{\mathcal{F}}, a_2^\dagger \mid 0 >_{\mathcal{F}}, a_1^\dagger a_2^\dagger \mid 0 >_{\mathcal{F}}\} \\ \mathcal{V}_{\mathcal{F}^*} &= \{\mathrm{Id} \mid 0 >_{\mathcal{F}^*}, a_1 \mid 0 >_{\mathcal{F}^*}, a_2 \mid 0 >_{\mathcal{F}^*}, a_1 a_2 \mid 0 >_{\mathcal{F}^*}\}. \end{aligned} \tag{8-73}$$

The line which connects these two edges can be reached by Bogoliubov-Valatin transformations. These states are usually employed in BCS theory for condensates.

Quasi free states are defined to have no higher correlations, i.e. there exists a transformation into a free theory, see [20, 21]. We can ask, which states in our plane do not possess higher correlations ($\kappa_n = 0$, for all $n > 1$, κ_n is defined below). Hence we have to assert that

$$\begin{aligned} \kappa_1(a_\alpha a_\beta^\dagger) &= \omega_{\nu w}(a_\alpha a_\beta^\dagger) \ =\ \nu \\ 0\ =\ \kappa_2(a_{\alpha_1} a_{\alpha_2} a_{\beta_1}^\dagger a_{\beta_2}^\dagger) &= \omega_{\nu w}(a_{\alpha_1} a_{\alpha_2} a_{\beta_1}^\dagger a_{\beta_2}^\dagger) \\ &\quad + \omega_{\nu w}(a_{\alpha_1} a_{\beta_1}^\dagger)\omega_{\nu w}(a_{\alpha_2} a_{\beta_2}^\dagger) \\ &\quad - \omega_{\nu w}(a_{\alpha_1} a_{\beta_2}^\dagger)\omega_{\nu w}(a_{\alpha_2} a_{\beta_1}^\dagger) \\ &= w - \nu^2 \end{aligned} \tag{8-74}$$

holds. From $\kappa_2 = 0$ we find a parabola in our diagram, which connects the Fock and dual Fock states and shows that these states are quasi free too. Having no higher correlations means that there is no interaction, hence these states build a border between regions having interactions of possibly different type, e.g. attracting or rejecting. Since we know that the line which connects Fock and dual Fock states is related to BCS theory, and since one has a condensate due to an attractive interaction, we may address the area between the parabola of quasi free states and the line of Bogoliubov-Valatin states as the condensate area.

Since every positive state can be written as a convex combination of extremal states, it remains to discuss the third edge of the triangle, which we call with Kerschner ‘edge‘-state and denote it by $\omega_{\mathcal{E}}$. We know that this state is at the position $\nu = 1/2$, $w = 0$. It is easy to see that this condition leads to the following 8-dimensional space

$$\begin{aligned}
\omega_{\mathcal{E}} &= \omega_{\nu w}\Big|_{\nu=1/2\, w=0} = \omega_{1/2\,0} \\
\mathcal{V}_{\mathcal{E}} &= \{ \mid 0 >_{\mathcal{E}}, a_1 \mid 0 >_{\mathcal{E}}, a_2 \mid 0 >_{\mathcal{E}}, a_1^{\dagger} \mid 0 >_{\mathcal{E}}, a_2^{\dagger} \mid 0 >_{\mathcal{E}}, \\
&\quad \frac{1}{2}(a_1 a_1^{\dagger} - a_2 a_2^{\dagger}) \mid 0 >_{\mathcal{E}}, a_1 a_2^{\dagger} \mid 0 >_{\mathcal{E}}, a_2 a_1^{\dagger} \mid 0 >_{\mathcal{E}} \}.
\end{aligned} \tag{8-75}$$

It is remarkable that in this set a spin triplet occurs which is not present in the Fock or dual Fock space. Moreover, we find spin up and down particles and antiparticles (annihilators w.r.t. the Fock vacuum!).

If one derives a gap-equation, see Ref. [55], one notes that the discriminant is negative for states in the area between the parabola of quasi free states and the edge-state, which disallows two solutions. On the other hand, if one looks at states between the quasi free parabola and the Bogoliubov-Valatin states (left border line) one has two solutions and a gap. This gap can be related to the common energy gab of BCS theory.

Having discussed roughly the vacuum states which arise from $\epsilon^{\wedge}$ by Hopf algebraic means, especially by cliffordization, we close this comprehensive treatise.

However we want to remark that this is only the starting point into a new and exciting field, which we await to be fruitful for studies in various directions. Hopf gebras will help us to understand what quantization means geometrically, a new approach to renormalization is opened, the vacuum/state space structures of a theory can be explored, dynamics is related to states directly, which will have interesting consequences, and many more. We await to enter hopfish times and quantum Clifford algebras will play a major role.

Appendix A

CLIFFORD and BIGEBRA packages for Maple

A.1 Computer algebra and Mathematical physics

Computer algebra was a major tool to investigate the topics which have been presented in this work. We had the opportunity to state even some theorems which we proved in low dimensions by direct calculations. Of course, the strength of a Computer Algebra System (CAS) is not to prove general theorems, but to provide a general area to explore mathematics and physics in an *experimental* way. Moreover, a CAS can help to surmount difficulties which would not be tractable at the moment by analytical, algebraical or arithmetical methods. E.g. when we computed the antipode of a two dimensional Clifford bi-convolution algebra this took some hours of computing time on a present day state of the art computer with lots of RAM. Only after the solution is found, it is an easy task to check by hand, so not relying on the computer any more, that this is indeed the searched antipode. A much wider area is opened by the possibility to check own and other people's assertions and claims simply by evaluating them in special cases. While this cannot lead to a proof, many such assertions can be disproved. This leads at the end to a refinement of their formulations and eventually to an idea how to prove such mathematical assertions by generalizing the generic case. Also in this work, we had the opportunity to find out many shortcomings of statements found in the literature. As a prominent example may be recalled the distinction between interacting and non-interacting, i.e. connected and non-connected, Hopf gebras. A simple re-calculation of standard material led to the fact that a Clifford Hopf gebra cannot be connected which stems from the non-locality of the cliffordization. Seeing the problem was essential to come up with a solution.

We want to summarize the cutting edge points which were valuable to the present research and which will become for sure a common tool in research in future times.

- **Check Assertions:** If one has a prejudice that some assertion should be true in an algebraic setting, randomly chosen special cases can give confidence into such a belief. More boldly, a single counterexample can put down the whole business immediately. This might look distracting but saves a tremendous amount of work, since only such assertions remain for being proved which are already tested to some amount and have a particular chance to generalize to a theorem.

- **Computations:** It should not be underestimated how time consuming it is to evaluate lengthy computations. While the CAS cannot substitute a sever knowledge of the mathematics behind and a sound physical concept to work on, it can help to compute with much fewer errors than any calculation by hand can provide. Moreover, using a CAS one can reach areas which are un-tractable by hand-written calculations simply by its mere length.

- **Develop new Mathematics:** Since new mathematical tools are not shipped with a CAS, one has to develop ones own functionality as an add to the common features of such a system. E.g. Maple [92][1] comes already with a tremendous ability to deal with many parts of algebra, but it was not able to deal with Graßmann and Clifford algebras and Hopf gebras. The development of such a device was a major impulse to investigate the mathematical structure in great depth. In fact, if you can teach the mathematics to a computer you have really understood the case.

- **Experimental Mathematics:** Having the opportunity to deal with a CAS opens the field of experimental mathematics. This includes partly the other topics of this list, but should not underestimated in its own dynamics. Exploring mathematics by doing particular experiments justifying or deceasing *own* assumptions is of extreme value to be able to enter a field fast and in a secure and solid way. This leads immediately to the next item.

- **Teaching:** *Experimental Mathematics* may be regarded as an additional tool in teaching complicated mathematics. Students can see what type of behaviour some algebraic or physical structures have before the try to understand or perform on their own a proof to master finally the topic. The CAS enables dealing in a concrete way with mathematical structures. Visualisation, erasing of miss-conceptions, and allowing a neat approach to complicated technicalities have already boosted up the field of non-linear dynamics. This field enjoyed a renaissance after the advent of sufficiently fast computers to handle the numerics. However, CAS is much more valuable since it really develops the algebraic understanding of the mathematical subject.

The particular CAS we use here is Maple V rel 5.1. Perhaps any reasonable general such tool could be employed. However, the already existing package CLIFFORD, developed by Rafał Abłamowicz [2], which I had enjoyed to use for now a couple of years, was reason enough for this choice.

In the next section we will give some hints how CLIFFORD can be used for computations in Clifford algebra. However, since there is a valuable and well developed online help consisting

[1] Maple is a registered trademark of Maple Waterloo Software, see `http://www.maplesoft.com/`

of approx. 150 help-pages, we stay with those features which were actually used in this work and which were essential for the development and design of the BIGEBRA package. The latest version of CLIFFORD is Cliff5 (i.e. version 5). CLIFFORD will be developed jointly in future with Rafał Abłamowicz.

The section on the BIGEBRA package will describe in a very cursory way the essential features which have been used to establish the assertions and theorems stated in this works. Some proofs have been by "direct computation using CLIFFORD/BIGEBRA" and we feel responsible to exemplify the abilities of CLIFFORD/BIGEBRA to give some hints how this was established. Full confidence can however be obtained only by looking at the particular, sometimes long-winding, Maple worksheets containing the actual computations. BIGEBRA was developed in close cooperation jointly with Rafał Abłamowicz.

A.2 The CLIFFORD Package – rudiments of version 5

The CLIFFORD package was developed by Rafał Abłamowicz since 1996. It is available from his web-server at `http://math.tntech.edu/rafal/`. From version 5 onwards the package comes together with the additional BIGEBRA package and is developed jointly with the author. Since there is an extensive online documentation, included into the Maple online help system, with help-page for every function we give only a look-and-feel description of those functions which are needed later in the BIGEBRA examples.

To load the CLIFFORD package we simply type in the following command:

```
> restart:with(Cliff5):
```

This has loaded the package and offers now to perform calculations in Graßmann and Clifford algebras. First of all, let us show how to select a Clifford algebra and how to assemble a basis, particular, and general elements. Such elements will be called Clifford or Graßmann polynoms, Clifford or Graßmann monoms with or without a scalar pre-factor. We compute over general algebraic expressions dealing thus with Clifford or Graßmann modules. A colon suppresses the output of the command, while a semi-colon ends a statement and returns its output. The generators of the algebras are denoted as `e1,e2,e3,`$\ldots$`,ea,eb,`$\ldots$.

```
> dim_V:=2:                    ## set dim. of generating space
> B:=linalg[diag](1$dim_V);  ## diagonal Euclidean metric; $ short for seq.
```

$$B := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> bas:=cbasis(dim_V);          ## get a basis spanning the Algebra
```

$$bas := [Id,\ e1,\ e2,\ e1we2]$$

5
```
> p1:=e1we2;                ## notion for 'e1 wedge e2'
```

$$p1 := e1we2$$

6
```
> p2:=a*e1+b*e1we2-4*Id;    ## a Grassmann polynom, Id is the unit
```

$$p2 := a\, e1 + b\, e1we2 - 4\, Id$$

7
```
> p3:=x*eaweb+ec;           ## a Grassmann polynom with symbolic indices
```

$$p3 := x\, eaweb + ec$$

8
```
> X:=add(_X[i]*bas[i],i=1..2^dim_V); ## a general element
```

$$X := \_X_1\, Id + \_X_2\, e1 + \_X_3\, e2 + \_X_4\, e1we2$$

Since the wedge product $\wedge$ was already used internally for building the Graßmann basis, we start by exemplifying the usage of the wedge product.

9
```
> wedge(e1,e2);             ## wedge of e1 and e2
```

$$e1we2$$

10
```
> &w(e1,e2);                ## short form for wedge
```

$$e1we2$$

11
```
> e1 &w e2;                 ## infix form for wedge
```

$$e1we2$$

12
```
> &w(p1,p2);                ## wedge on particular elements
```

$$-4\, e1we2$$

```
> &w(X,X);                  ## square of a general element
```

$$\_X_1{}^2\, Id + 2\,\_X_2\,\_X_1\, e1 + 2\,\_X_3\,\_X_1\, e2 + 2\,\_X_4\,\_X_1\, e1we2$$

Given the Graßmann algebra as above, we have also contractions at our disposal. The contractions act w.r.t. the chosen bilinear form B, which could also be symbolic or unassigned at all. The (left) contraction acts as a graded derivation on the module generated by the above given basis. It also established the bilinear form. To manipulate Graßmann basis elements we need also a device to put them into a standard order, i.e. the function ‘`reorder`’ and a function which constitutes the grading, i.e. ‘`gradeinv`’. The eigenspace of `gradeinv` are exactly the even and odd elements.

```
> map(gradeinv,bas);               ## map means ‘apply to the list‘
```

$$[Id,\ -e1,\ -e2,\ e1we2]$$

```
> map(i->1/2*(i+gradeinv(i)),bas); ## even elements
```

$$[Id,\ 0,\ 0,\ e1we2]$$

```
> map(i->1/2*(i-gradeinv(i)),bas); ##  odd elements
```

$$[0,\ e1,\ e2,\ 0]$$

```
> linalg[matrix](dim_V,dim_V,(i,j)->LC(e.i,e.j));## contraction on vectors
```

$$\begin{bmatrix} Id & 0 \\ 0 & Id \end{bmatrix}$$

```
> ## derivation property, LC taken w.r.t. the unassigned bilinear form ‘K’
> LC(e1,e1we2,K)=LC(e1,e1,K) &w e2+gradeinv(e1)*LC(e1,e2,K);
```

$$K_{1,1}\, e2 - K_{1,2}\, e1 = K_{1,1}\, e2 - e1\, K_{1,2}\, Id$$

It is well know that the Clifford product of a 1-vector can be established as an endomorphism on the Graßmann basis underlying the Clifford algebra. Such a particular endomorphism is called a *Clifford map*. The Clifford product in CLIFFORD ver. 5 is however based on the Hopf algebraic process of Cliffordization.

```
> CliMap:=proc(x,u,B) LC(x,u,B)+wedge(x,u) end: ## the Clifford map
> CliMap(e1,Id,B);    ## contraction part is zero
```

$$e1$$

```
> CliMap(e1,e1,K);    ## wedge part is zero
```

$$K_{1,1}\, Id$$

```
> CliMap(e1,e2,K);    ## Clifford product w.r.t. the bilinear form 'K'
```

$$K_{1,2}\, Id + e1we2$$

```
> CliMap(e2,e1we2,B); ## action on a bi-vector
```

$$-e1$$

```
> cmul(e2,e1we2);     ## compare with the builtin Clifford product
```

$$-e1$$

Of course, the Clifford product has to be extended to a general first argument. This can be done by using the rules given in the main text. Since more features of CLIFFORD are explained in the following section which describes the BIGEBRA package, we end by exemplifying the `clisolve` facility. This function allows to solve equations in Graßmann and Clifford algebras either for particular elements and their coefficients or for arbitrary elements. We will show how to find idempotents. Remember that we had defined an arbitrary element X.

```
> X; ## general element; dim_V = 2
```

$$\_X_1\, Id + \_X_2\, e1 + \_X_3\, e2 + \_X_4\, e1we2$$

```
> sol:=map(allvalues,clisolve(cmul(X,X)-X,X));
```

$$\begin{aligned} sol := [0,\ Id,\ & \frac{1}{2}\, Id + \frac{1}{2}\sqrt{1+4\,\_X_4{}^2}\; e1 + \_X_4\, e1we2,\ \frac{1}{2}\, Id - \frac{1}{2}\sqrt{1+4\,\_X_4{}^2}\; e1 + \_X_4\, e1we2, \\ & \frac{1}{2}\, Id + \_X_2\, e1 + \frac{1}{2}\sqrt{-4\,\_X_2{}^2 + 4\,\_X_4{}^2 + 1}\; e2 + \_X_4\, e1we2, \\ & \frac{1}{2}\, Id + \_X_2\, e1 - \frac{1}{2}\sqrt{-4\,\_X_2{}^2 + 4\,\_X_4{}^2 + 1}\; e2 + \_X_4\, e1we2] \end{aligned}$$

```
> ## re-compute the equation to check for correctness
> sol_square:=map(i->clicollect(simplify(cmul(i,i)-i)),sol);
```

$$sol\_square := [0, 0, 0, 0, 0, 0]$$

All functions come with well developed help-pages. They can be reached by typing ?function at the Maple commandline or searching the help of Maple. A general help-page for the entire package and its sub-packages is available by typing ?Clifford[intro]. A general introduction to Maple and its programming facilities may be found e.g. in [138].

A.3 The BIGEBRA Package

This appendix provides only a very basic look-and-feel explanation of the BIGEBRA package. The online documentation of BIGEBRA comes with over 100 printed pages and should be consulted as reference. However, we felt it necessary to exhibit BIGEBRA's abilities here, since it was used to prove some statements in the text.

The BIGEBRA package (version 0.16) loads automatically the CLIFFORD package since the latter package is internally needed. We suppress the startup messages by setting _SILENT to $true$.

```
> restart:_CLIENV[_SILENT]:=true:with(Bigebra):
```

```
Warning, new definition for drop_t
Warning, new definition for gco_d_monom
Warning, new definition for gco_monom
Warning, new definition for init
```

The particular functions of BIGEBRA are described below very shortly to give an overview. For detailed help-pages and much more detailed examples use the Maple online help by typing ?Bigebra,<function>.

A.3.1 &cco – Clifford co-product

The internal computation of the Clifford co-product is done by Rota-Stein co-cliffordization as explained in the main text. The Clifford co-product has therefore to be initialized before the first usage, since it needs internally the Clifford co-product of the unit element, i.e. the 'cap' tangle. Furthermore one needs also a co-scalar product which is stored in the matrix BI (or left undefined), the dimension of the base space, defined in dim_V, can range between 1 and 9. We have to set:

```
> dim_V:=2:
> BI:=linalg[matrix](dim_V,dim_V,[a,b,c,d]);
```

$$BI := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

```
> make_BI_Id():
> &cco(e1);
```

$(Id \mathbin{\&\mathrm{t}} e1) - b\,(e1 \mathbin{\&\mathrm{t}} e1we2) - d\,(e2 \mathbin{\&\mathrm{t}} e1we2) + (e1 \mathbin{\&\mathrm{t}} Id) + c\,(e1we2 \mathbin{\&\mathrm{t}} e1) + d\,(e1we2 \mathbin{\&\mathrm{t}} e2)$

The most remarkable fact is that the Clifford co-product of the unit element `Id` is not `&t(Id,Id)` but

```
> &cco(Id);
```

$(Id \mathbin{\&\mathrm{t}} Id) + a\,(e1 \mathbin{\&\mathrm{t}} e1) + c\,(e2 \mathbin{\&\mathrm{t}} e1) + b\,(e1 \mathbin{\&\mathrm{t}} e2) + d\,(e2 \mathbin{\&\mathrm{t}} e2) + (c\,b - d\,a)\,(e1we2 \mathbin{\&\mathrm{t}} e1we2)$

The Clifford co-product is however co-associative.

A.3.2 `&gco` – Graßmann co-product

The Graßmann co-product is the basic function of the BIGEBRA package, since the Clifford co-product is derived by the process of co-cliffordization. It turns out that the Graßmann co-product is a combinatorial function on the index set of Graßmann multi-vectors, this is used in the package to get a fast evaluation of this function. The Graßmann co-product is that of a connected and augmented co-algebra, which we called non-interacting Hopf gebra in the main text.

```
> &gco(Id);    ## this is as expected
```

$$Id \mathbin{\&\mathrm{t}} Id$$

```
> &gco(e1);
```

$$(Id \mathbin{\&\mathrm{t}} e1) + (e1 \mathbin{\&\mathrm{t}} Id)$$

```
> &gco(e1we2); ## sum over splits
```

$$(\mathit{Id} \mathbin{\&\mathrm{t}} \mathit{e1we2}) + (\mathit{e1} \mathbin{\&\mathrm{t}} \mathit{e2}) - (\mathit{e2} \mathbin{\&\mathrm{t}} \mathit{e1}) + (\mathit{e1we2} \mathbin{\&\mathrm{t}} \mathit{Id})$$

Note that in the last case the sum is over all splits which are compatible with the permutation symmetry of the factors. The signs are such that multiplying back gives *for each term* the original input. Hence we get two to the power of the grade of the element as a prefactor:

```
> eval(subs(`&t`=wedge,[op(%)]));
```

$$[\mathit{e1we2},\ \mathit{e1we2},\ \mathit{e1we2},\ \mathit{e1we2}]$$

```
> eval(`+`(op(%)));
```

$$4\ \mathit{e1we2}$$

A.3.3 `&gco_d` – dotted Graßmann co-product

The dotted Graßmann co-product is taken with respect to a different filtration of the Graßmann algebra under consideration. This different filtration is represented by the dotted wedge basis built w.r.t the dotted wedge product $\dot{\wedge}$. The dotted Graßmann co-product is a wrapper function which translates the wedge basis elements into the dotted wedge basis ones, computes there the regular Graßmann co-product and transforms back the tensor product into the undotted basis. For examples see the online help of BIGEBRA.

A.3.4 `&gpl_co` – Graßmann Plücker co-product

The Graßmann-Plücker co-product evaluates the co-product w.r.t. the `meet` (resp. `&v`) product of hyperplanes since it can be shown that the meet is an exterior product for hyperplanes. If we represent hyperplanes using Plücker coordinates, we can ask for a co-product on these Plücker coordinatized hyperplanes, which is in fact related to the wedge product of the points. For examples see the online help of BIGEBRA.

A.3.5 `&map` – maps products onto tensor slots

The `&map` function extends product to be able to act on tensors. For instance one wants to wedge or Clifford multiply a tensor, say `&t(e1,e2we3,e1we2)`, in two adjacent slots of the tensor. This is achieved as

```
> dim_V:=4:
> &map(&t(e1,e2we3,e1we4),2,wedge);
```

$$e1 \,\&\mathrm{t}\; e1we2we3we4$$

43
```
> &map(&t(e1,e2we3,e1we4),1,cmul);
```

$$(e1we2we3 \,\&\mathrm{t}\; e1we4) + B_{1,2}\,(e3 \,\&\mathrm{t}\; e1we4) - B_{1,3}\,(e2 \,\&\mathrm{t}\; e1we4)$$

Any $2 \to 1$ mapping can be applied to tensors by this device. As most of the BIGEBRA and CLIFFORD functions this is a multilinear mapping.

A.3.6 `&t` – tensor product

The tensor product is a basic feature of the BIGEBRA package. The tensor product is an unevaluated product which is multilinear over any Maple expression which is not a CLIFFORD basis element. That is we are able to compute over Clifford modules. However, re-defining the Clifford type `type/cliscalar` one can change the behaviour. A few examples are

44
```
> &t(a*e1,3*e2+5*e3);
```

$$3\,a\,(e1 \,\&\mathrm{t}\; e2) + 5\,a\,(e1 \,\&\mathrm{t}\; e3)$$

45
```
> &t(e1,sin(x)*e2,e1we2);
```

$$\sin(x)\,\&\mathrm{t}(e1,\ e2,\ e1we2)$$

46
```
> &t(e1we2*z,-e3/z+t*e4,e2/t);
```

$$-\frac{\&\mathrm{t}(e1we2,\ e3,\ e2)}{t} + z\,\&\mathrm{t}(e1we2,\ e4,\ e2)$$

The tensor product allows studying decomposition and periodicity theorems. One can handle multi-particle Clifford algebra, compute in different Clifford algebras, e.g. different bilinear or quadratic forms, and is able to investigate tangles of Graßmann Hopf gebras and Clifford convolution algebras. A computation of a Graßmann or Clifford antipode would be impossible without this device. Moreover, also more geometric notions as the `meet` or `&v` (vee) product benefit from this structure.

A.3.7 `&v` – vee-product, i.e. meet

The meet or vee-product computes the join of two extensors. It constitutes an exterior product on its own right, but on hyperplanes, not on points. If hyperplanes are identified which the duals of points, which needs a correlation and introduces a bilinear form, a complete dual approach to the Graßmann-Cayley algebra and its deformed structure the Clifford convolution algebra is obtained. A few examples are:

```
> dim_V:=3:B:='B': ## unassign B
> meet(e1we2,e2we3),&v(e1we2,e2we3); ## meet and &v are the same
```

$$-e2,\ -e2$$

```
> &v(e1we2+e2we3,e2we3+e1we3);        ## acts on polynoms too
```

$$-e1 - e2 + e3$$

Note that the meet introduces signs and it is the oriented meet of the support of the extensor which describes the linear subspace. Of course a geometrical meaning of polynomial such objects is not obvious, but the meet nevertheless inherits linearity from its construction. The meet is calculated using the Peano `bracket` and the co-product as

$$\mathrm{meet}(x, y) = x_{(1)}[y, x_{(2)}] = [y_{(1)}, x]y_{(2)}$$

where the order of factors is important. The bracket can be understood in hopfish terms too.

A.3.8 `bracket` – the Peano bracket

The Peano bracket and Peano algebra was introduced by Rota et al. [43, 11] and called in the first paper Cayley algebra. However, Peano introduced the bracket as a device to define Graßmann's regressive product in dimension three, see [105]. We showed in the main text that the Peano bracket can be derived using a non-trivial integral of the Graßmann Hopf gebra.

$$[x, y] = h(x \wedge y)$$

where $h(x) : \bigwedge V \to \Bbbk$ is a non-trivial integral. In the case of the Graßmann Hopf gebra this is the projection onto the highest grade element. BIGEBRA needs thus no bilinear form to define the bracket but only a maximal dimension. The `bracket` function takes any number of arguments, wedges them together and projects onto the highest grade, e.g.

```
> dim_V:=3:
> bracket(e1we2we3),bracket(e1,e2,e3);
```

$$1,\ 1$$

```
> dim_V:=4:
> bracket(e1we2,e2we3we4); ##   0 expected
```

$$0$$

```
> bracket(a*e1we2,b*e3we4);## a*b expected
```

$$a\,b$$

A.3.9 `contract` – contraction of tensor slots

Given a tensor with at least two slots, contract allows to map a $2 \to 0$ mapping onto adjacent such slots. The tensor elements can be seen as vectors or co-vectors, so we have in fact 4 types of contractions.

```
> contract(&t(e1,e1,e2),1,EV);          ## evaluation on slots 1,2
```

$$\&\mathrm{t}(e2)$$

```
> contract(&t(e1,e1we2,e3we4),2,bracket); ## bracket on slots 2,3
```

$$\&\mathrm{t}(e1)$$

A.3.10 `define` – Maple define, patched

The `define` facility of Maple turned out to be not very useful for defining multilinear associative functions. It showed up to compute wrong results and was not designed to handle an arbitrary base ring. BIGEBRA patches define so that `type/cliscalar` is used for scalars and that any function defined with define like `define(‘&r‘,flat,multilinear)` to be associative, i.e. flat and multilinear. For further information see the online help-page of BIGEBRA.

A.3.11 `drop_t` – drops tensor signs

This is a helper function to drop the tensor sign `&t` from Clifford expressions, i.e. tensors of rank one. For technical reasons the tensor sign is not automatically dropped.

```
> drop_t(&t(a*e1+b*e1we2));
```

$$a\,e1 + b\,e1we2$$

A.3.12 EV – evaluation map

The evaluation map is given by the action of co-vectors on vectors acting in the natural way. If a canonical co-basis θ^a is defined, one finds $\theta^a(e_b) = \delta^a_b$ where δ is the Kronecker symbol. The user has to take care in which tensor slot the co-vectors reside, since they are, unfortunately, displayed by the same basis symbols $eiwej$ etc. The evaluation map acts on any multivector polynom in $\bigwedge V$.

```
> EV(e1,a*Id+b*e1+c*e2+d*e1we2);   ## b expected
```

$$b$$

```
> EV(e1we2,e1we2),EV(e1we2,e2we3); ## 1,0 expected
```

$$1,\ 0$$

A.3.13 gantipode – Graßmann antipode

The Graßmann antipode is the antipode of the Graßmann Hopf gebra. The most remarkable fact is that this antipode map is equivalent to the main involution of a Clifford algebra of the same space or the main involution of the Graßmann algebra

```
> dim_V:=3:
> bas:=cbasis(dim_V);
```

$$bas := [Id,\ e1,\ e2,\ e3,\ e1we2,\ e1we3,\ e2we3,\ e1we2we3]$$

```
> map(gantipode,bas);
```

$$[Id,\ -e1,\ -e2,\ -e3,\ e1we2,\ e1we3,\ e2we3,\ -e1we2we3]$$

```
> map(gradeinv,bas);
```

$$[Id,\ -e1,\ -e2,\ -e3,\ e1we2,\ e1we3,\ e2we3,\ -e1we2we3]$$

A.3.14 `gco_unit` – **Graßmann co-unit**

Since the co-gebra structure is obtained by categorical duality, the Graßmann co-gebra possesses a co-unit. This can be exemplified as follows:

```
64 > _X:=add(X[i]*bas[i],i=1..2^dim_V);    ## arbitrary element
```

$$\_X := X_1\, \mathit{Id} + X_2\, \mathit{e1} + X_3\, \mathit{e2} + X_4\, \mathit{e3} + X_5\, \mathit{e1we2} + X_6\, \mathit{e1we3} + X_7\, \mathit{e2we3} + X_8\, \mathit{e1we2we3}$$

```
65 > simplify(drop_t(gco_unit(&gco(_X),1)) - _X); ## 0 expected
```

$$0$$

```
66 > simplify(drop_t(gco_unit(&gco(_X),2)) - _X); ## 0 expected
```

$$0$$

A.3.15 `gswitch` – **graded (i.e. Graßmann) switch**

The graded switch is the natural switch of the Graßmann Hopf gebra. It is not the generic switch of a Clifford algebra if the bilinear form is not identical zero. The graded switch swaps two adjacent factors of a tensor and counts the minus signs arising from the reordering of the factors.

```
67 > gswitch(&t(e1,e2,e3we4),1); ## - expected
```

$$-\&\mathrm{t}(\mathit{e2},\, \mathit{e1},\, \mathit{e3we4})$$

```
68 > gswitch(&t(e1,e2,e3we4),2); ## + expected
```

$$\&\mathrm{t}(\mathit{e1},\, \mathit{e3we4},\, \mathit{e2})$$

A.3.16 `help` – **main help-page of BIGEBRA package**

This is not a function of the package, but the main help-page of the BIGEBRA package. It can be accessed in a Maple session by typing `?Bigebra,help`. The main help-page gives an alphabetic listing of BIGEBRA functions, links it to CLIFFORD and provides some literature from which place some of the algorithms and mathematics have been taken. The reader is urged to look up this page.

A.3.17 `init` – init procedure

BIGEBRA needs a tricky init procedure to patch load the package and patch the Maple `define` function. Init loads BIGEBRA, then the tensor product `&t` is defined which loads the define code into the session. Then BIGEBRA is loaded a second time to overwrite in the memory the unsuited parts of define. `Init` loads CLIFFORD, i.e. Cliff5, if it was not already loaded.

A.3.18 `linop/linop2` – action of a linear operator on a Clifford polynom

Since we have been interested in tangle equations like the definition of the antipode. The action of certain operators on a tensor slot is therefore necessary. Sometimes it is useful to have matrix representations of such operators and `linop` provides this facility. `linop2` is the same function which acts however on two adjacent tensor slots, hence we have

$$\mathrm{linop} \in \mathrm{End} \bigwedge V$$

$$\mathrm{linop2} \in \mathrm{End} \bigwedge V \otimes \bigwedge V$$

A.3.19 `make_BI_Id` – cup tangle need for `&cco`

This function computes the cap tangle for a certain co-scalar product either unassigned or defined as a matrix named `BI`. See either `&cco` above or the online help-page of BIGEBRA.

A.3.20 `mapop/mapop2` – action of an operator on a tensor slot

While `linop(2)` defines a linear operator as an endomorphism on $\bigwedge V$ seen as linear space. The function `mapop(2)` allows to apply these operators to any tensor slot of a tensor or to any two adjacent tensor slots. For some example and the usage see the help-page of BIGEBRA.

A.3.21 `meet` – same as `&v` (vee-product)

The meet is a synonym for the `&v` (vee-) product. However, in the BIGEBRA package the meet and vee-products are computed differently, we have

$$meet(x,y) = x_{(1)}[y, x_{(2)}]$$

while

$$\&v(x,y) = [y_{(1)}, x]y_{(2)}]$$

This allows to check that both definitions are equivalent. This computation can be found, together with many geometric applications and some benchmarks in the online help-page for the `meet` in the BIGEBRA package.

A.3.22 `pairing` – A pairing w.r.t. a bilinear form

The pairing is a decorated cup tangle, where the decoration describes the bilinear form used to convert one element into a co-vector, i.e. a scalar product. The pairing s graded and can be defined as follows

$$\langle x \mid y \rangle = \begin{cases} \pm\det(\langle x_i \mid y_j \rangle) & \text{if grade } x = \text{grade } y \\ 0 & \text{otherwise} \end{cases}$$

where x, y are extensors of $\bigwedge V$ and the pairing is extended by bilinearity. For explicite examples see the online help-page of the BIGEBRA package.

A.3.23 `peek` – extract a tensor slot

This is a technical function used mostly internally to be able to access certain tensor slots. For explicite examples and the correct syntax see the online help-page of the BIGEBRA package.

A.3.24 `poke` – insert a tensor slot

This is a technical function used mostly internally to be able to insert Clifford elements as new tensor slots in an arbitrary tensor polynomial. For explicite examples and the correct syntax see the online help-page of the BIGEBRA package.

A.3.25 `remove_eq` – removes tautological equations

This is a technical function used mostly internally. It drops tautological equations in a set of equations. For explicite examples see the online help-page of the BIGEBRA package.

A.3.26 `switch` – ungraded switch

The switch simply swaps adjacent tensor slots, no sign is computed.

69
```
> switch(&t(e1,e2,e3we4),1);
```

$$\&\mathrm{t}(e2,\ e1,\ e3we4)$$

70
```
> switch(&t(e1,e2,e3we4),2);
```

$$\&\mathrm{t}(e1,\ e3we4,\ e2)$$

A.3.27 `tcollect` – collects w.r.t. the tensor basis

This is a function which is needed to customise the output of some BIGEBRA functions for inputting it into other such functions. Furthermore it allows a better comparison of tensor polynomials. For explicite examples see the online help-page of the BIGEBRA package.

A.3.28 `tsolve1` – tangle solver

The tangle solver is an extension of the CLIFFORD function `clisolve`. It allows to solve for *endomorphisms* acting in $n \to 1$ tangles, therefore the name. Most of the axioms and definitions of Graßmann Hopf gebras and Clifford bi-convolution algebras are of this type. The online help-page for `tsolve1` comes up with explicite computations of the unit for Graßmann convolution, the Graßmann antipode and some facts about integrals in Graßmann and Clifford bi-convolutions. For explicite examples see the online help-page of the BIGEBRA package.

A.3.29 `VERSION` – shows the version of the package

This command is issued as `VERSION();` and returns some information about the release of the BIGEBRA package.

A.3.30 `type/tensorbasmonom` – new Maple type

To be able to facilitate symbolic computations Maple provides a type checking system. BIGEBRA as CLIFFORD use this device and define some new types extending this mechanism. A tensorbasmonom is any expression which is an extensor without any prefactor, e.g.

```
71 > type(&t(e1,e2,e3),tensorbasmonom);   ## true  expected
```

$$true$$

```
72 > type(a*&t(e1,e2),tensorbasmonom);    ## false expected
```

$$false$$

```
73 > type(&t(e1)+&t(e2),tensorbasmonom);  ## false expected
```

$$false$$

```
74 > type(a*sin(x)*e1we3,tensorbasmonom); ## false expected
```

$$false$$

A.3.31 `type/tensormonom` – new Maple type

A tensormonom is a tensorbasmonom possibly having a prefactor from the ring the tensor product is built over. This type is inclusive in that way that a tensorbasmonom is also considered to be a tensormonom.

```
75 > type(&t(e1,e2,e3),tensormonom);   ## true  expected
```

$$true$$

```
76 > type(a*&t(e1,e2),tensormonom);    ## true  expected
```

$$true$$

```
77 > type(&t(e1)+&t(e2),tensormonom);  ## false expected
```

$$false$$

```
78 > type(a*sin(x)*e1we3,tensormonom); ## false expected
```

$$false$$

A.3.32 `type/tensorpolynom` – new Maple type

A tensor polynom is a sum of tensormonoms. This type is also inclusive.

```
79 > type(&t(e1,e2,e3),tensorpolynom);   ## true  expected
```

$$true$$

```
80 > type(a*&t(e1,e2),tensorpolynom);    ## true  expected
```

$$true$$

```
81 > type(&t(e1)+&t(e2),tensorpolynom);  ## true  expected
```

$$true$$

```
82 > type(a*sin(x)*e1we3,tensorpolynom); ## false expected
```

$$false$$

Bibliography

[1] Eiichi Abe. *Hopf Algebras*. Cambridge University Press, Cambridge, 1980. (transl. of the 1977 Japanese edition).

[2] Rafał Abłamowicz and Bertfried Fauser. *CLIFFORD*. Tennessee Technological University, `http://math.tntech.edu/rafal/`, 1996-2002. Maple V program package.

[3] Rafał Abłamowicz and Bertfried Fauser. *BIGEBRA*. Tennessee Technological University and University of Konstanz, `http://math.tntech.edu/rafal/`, 1999-2002. Maple V program package.

[4] Rafał Abłamowicz and Bertfried Fauser, editors. *Clifford Algebras and their Applications in Mathematical Physics*, number PPh 18, Boston, 2000. Birkhäuser.

[5] Rafał Abłamowicz and Bertfried Fauser. Hecke algebra representations by q-Young idempotents. In Rafał Abłamowicz and Bertfried Fauser, editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 245–268, Boston, 2000. Birkhäuser. math.QA/9908062.

[6] Rafał Abłamowicz and Bertfried Fauser. Efficient algorithms for Clifford products using Hopf gebra methods. 2002. in preparation.

[7] Rafał Abłamowicz and Bertfried Fauser. Efficient algorithms for meet and join in Graßmann-Cayley algebras. 2002. in preparation.

[8] J.L. Anderson. Green's functions in quantum electrodynamics. *Phys. Rev.*, 94:703–711, 1954.

[9] Friedrich Bachmann. *Aufbau der Geometrie aus dem Spiegelungsbegriff*. Springer-Verlag, Berlin, 1973. [first ed. 1959].

[10] Friedrich Bachmann. *Ebene Spiegelungsgeometrie*. BI Wissenschaftsverlag, Mannheim, 1989.

[11] Marilena Barnabei, Andrea Brini, and Gian-Carlo Rota. On the exterior calculus of invariant theory. *Journal of Algebra*, 96:120–160, 1985.

[12] Rodney J. Baxter. *Exactly solved models in statistical mechanics*. Academic Press, London, 1984. (2nd editon).

[13] Eduardo Bayro Corrochano and Garret Sobzcyk, editors. *Geometric algbera with applications in science and engineering*, Boston, MA, 2001. Birkhäuser.

[14] Ian M. Benn and Robin W. Tucker. *An introduction to spinors and geometry with applications in physics*. Hilger, Bristol, 1987.

[15] Walter Benz. *Vorlesungen über Geometrie der Algebren : Geometrien von Möbius, Laguerre-Lie, Minkowski in einheitlicher und grundlagengeometrischer Behandlung*. Springer-Verlag, Berlin, 1973. Die Grundlehren der mathematischen Wissenschaften, Bd. 197.

[16] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groups de Lie compacts. *Annals of Mathematics*, 57:115–207, 1957.

[17] Thomas Borne, Georges Lochak, and Harald Stumpf. *Nonperturbative Quantum Field Theory and the Structure of Matter*. Kluwer Academic Publishers, Dordrecht, 2001.

[18] Nicolas Bourbaki. *Elements of Mathematics: Algebra I – Chapters 1–3*. Springer-Verlag, Berlin, 1989.

[19] F. Brackx, R. Delanghe, and H. Serras, editors. *Clifford Algebras and their Applications in Mathematical Physics*, Dordrecht, 1993. Kluwer Academic Publishers. Deinze 1993.

[20] O. Bratelli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics*, volume I. Springer, Berlin, 1979.

[21] O. Bratelli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics*, volume II. Springer, Berlin, 1981.

[22] Chrisitan Brouder. A quantum field algebra. *J. Phys. A: Math. Gen. submitted*, 2002. math-ph/0201033.

[23] Christian Brouder. Noncommutative renormalization for massless QED. *preprint*, pages 1–51, 2000. hep-th/0011161.

[24] Christian Brouder. On the trees of quantum fields. *Eur. Phys. J.*, C(12):535–549, 2000. hep-th/9906111.

[25] Christian Brouder. Runge–Kutta methods and renormalization. *Eur. Phys. J.*, C(12):521–534, 2000. hep-th/9904014.

[26] John Browne. *Grassmann Algebra : Exploring applications of extended vector algebra with Mathematica*. Melbourne, Australia, 2001. (incomplete draft, available form `http://www.ses.swin.edu.au/homes/browne/`).

[27] Paolo Budinich and Andrzej Trautmann. *The Spinorial Chessboard*. Springer-Verlag, Berlin, 1988. Trieste Notes in Physics.

[28] J. C. Butcher. An algebraic theory of integration. *Mathematics of Computation*, 26(117):79–106, 1972.

[29] J. C. Butcher. A convergence criterion for a class of integration methods. *Mathematics of Computation*, 26(117):107–117, 1972.

[30] Eduardo R. Caianiello. *Combinatorics and renormalization in quantum field theory*. Benjamin, Mass., 1973.

[31] Claude Chevalley. *The Algebraic Theory of Spinors and Clifford Algebras*. Springer-Verlag, Berlin, 1997. Collected Works Vol. 2, Pierre Cartier, Catherine Chevalley Eds.

[32] J.S.R. Chisholm and A.K. Common, editors. *Clifford Algebras and their Applications in Mathematical Physics*, Dordrecht, 1986. Reidel. Canterbury 1985.

[33] Alain Connes and Dirk Kreimer. Lessons from quantum field theory. *Lett. Math. Phys.*, 48:85–96, 1999. hep-th/9904044.

[34] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem. *Commun. Math. Phys.*, 210:249–273, 2000.

[35] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann-Hilbert problem II: the β function, diffeomorphisms and the renormalization group. *Commun. Math. Phys.*, 216:215–241, 2001.

[36] Oliver Conradt. *Mathematical physics in space and counterspace*. PhD thesis, University of Basel, Basel, Switzerland, August 2000.

[37] Oliver Conradt. Mechanics in space and counterspace. *Journal of Mathematical Physics*, 41(10):6995–7028, 2000.

[38] Oliver Conradt. The principle of duality in Clifford algebra and projective geometry. In R. Abłamowicz and B. Fauser, editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 157–193, Boston, 2000. Birkhäuser.

[39] Michael J. Crowe. *A history of vector analysis : The evolution of the idea of a vectorial system*. Number i–xvii, 1. Dower Publications, Inc., New York, 1994. (first ed. 1967, Univ. of Notre Dame).

[40] Albert Crumeyrolle. *Orthogonal and Symplectic Clifford Algebra – Spinor Structures*. Kluwer Academic Publishers, Dordrecht, 1990.

[41] John de Pills. Grassmann algebras as Hilbert space. *Journal of Algebra*, 10:485–500, 1968.

[42] V. Dietrich, K. Habetha, and G. Jank, editors. *Clifford Algebras and their Applications in Mathematical Physics*, Dordrecht, 1998. Kluwer Academic Publishers. Aachen 1996.

[43] Peter Doubilet, Gian-Carlo Rota, and Joel Stein. On the foundations of combinatorial theory: IX Combinatorial methods in invariant theory. *Studies in Applied Mathematics*, LIII(3):185–216, September 1974.

[44] V.G. Drinfeld. Quantum groups. In *Proc. I.C.M.*, pages 798–820, Berkeley, 1987. Providence, RI: AMS.

[45] B. Eckmann. Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition der quadratischen Formen. *Comentarii Mathematici Helvetici*, 15:358–366, 1943.

[46] H. Epstein and V. Glaser. The role of locality in perturbation theory. *Ann. Inst. Henri Poincaré*, 19:211–295, 1973.

[47] Bertfried Fauser. *Clifford-algebraische Formulierung und Regularität in der Quantenfeldtheorie*. PhD thesis, Universität Tübingen, Tübingen, January 1996.

[48] Bertfried Fauser. Vertex normalordering as a consequence of nonsymmetric bilinear forms in Clifford algebras. *Journal of Mathematical Physics*, 37(1):72–83, 1996. hep-th/9504055.

[49] Bertfried Fauser. Dirac theory from a field theoretic point of view. In Dietrich V., K. Habetha, and G. Jank, editors, *Clifford algebras and their applications in mathematical physics*, pages 89–107, Dordrecht, 1998. Kluwer Academic Publishers. Aachen 1996.

[50] Bertfried Fauser. On an easy transition from operator dynamics to generating functionals by Clifford algebras. *Journal of Mathematical Physics*, 39(9):4928–4947, 1998. hep-th/9710186.

[51] Bertfried Fauser. Hecke algebra representations within Clifford geometric algebras of multivectors. *Journal of Physics A: Mathematical and General*, 32:1919–1936, 1999. q-alg/9710020.

[52] Bertfried Fauser. On the relation of Clifford-Lipschitz groups to q-symmetric groups. In S.P. Corney, R. Delbourgo, and P.D. Jarvis, editors, *GROUP 22, proc.*, pages 413–417, Cambridge, MA., 1999. International Press.

[53] Bertfried Fauser. On the relation of Manin's quantum plane and quantum Clifford algebras. *Czechoslovak Journal of Physics*, 50(11):1221–1228, November 2000. math.QA/0007137.

[54] Bertfried Fauser. Vertex functions and generalized normal-ordering by triple systems in non-linear spinor field models. *Advances in Applied Clifford Algebras*, 10(2):173–192, 2000. hep-th/9611069.

[55] Bertfried Fauser. Clifford geometric parameterization of inequivalent vacua. *Mathematical Methods in the Applied Sciences*, 24:885–912, 2001. hep-th/9710047v2.

[56] Bertfried Fauser. On the Hopf-algebraic origin of Wick normal-ordering. *Journal of Physics A: Mathematical and General*, 34:105–115, 2001. hep-th/0007032.

[57] Bertfried Fauser and Rafał Abłamowicz. On the decomposition of Clifford algebras of arbitrary bilinear form. In Rafał Abłamowicz and Bertfried Fauser, editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 341–366, Boston, 2000. Birkhäuser. math.QA/9911180.

[58] Bertfried Fauser and Heinz Dehnen. Quantum differential forms. 2002. in preparation.

[59] Bertfried Fauser and Zbigniew Oziewicz. Clifford Hopf gebra for two dimensional space. *Miscellanea Algebraicae*, 2(1):31–42, 2001. math.QA/0011263.

[60] Bertfried Fauser and Harald Stumpf. Positronium as an example of algebraic composite calculations. In J. Keller and Z. Oziewicz, editors, *The Theory of the Electron*, pages 399–418, Mexico, 1997. UNAM. Adv. in Appl. Clifford Alg. 7(suppl.), hep-th/9510193.

[61] William Fulton. *Young Tableaux*. Cambridge University Press, Cambridge, 1999. (first published 1997).

[62] Izrail' M. Gel'fand. *Kommutative normierte Algebren*. Dt. Verl. d. Wiss., Berlin, 1964.

[63] Hermann Graßmann. *Die Ausdehnungslehre; Vollständig und in strenger Form bearbeitet*. Verlag von Th. Chr. Fr. Enslin (Adolph Enslin), Berlin, 1862.

[64] Hermann Graßmann. *Die lineale Ausdehnungslehre; ein neuer Zweig der Mathematik*. Verlag Otto Wigand, Leipzig, 1878. (reprint of the original 1844 edition, with anotations of Graßmann).

[65] Gebhard Grimm. *Elektroschwache Standardtheorie als effektive Dynamik von Bindungszuständen im Rahmen einer nichtlinearen Spinortheorie*. PhD thesis, Universität Tübingen, Tübingen, 1994.

[66] Frank D. Grosshans, Gian-Carlo Rota, and Joel A. Stein. Invariant theory and superalgebras. In *conference board of the mathematical sciences regional conference series in mathematics, number 69*, pages i–xxi,1–80. American Mathematical Society, 1987.

[67] Frank Hausser and Florian Nill. Integral theory for quasi-Hopf algebras. 1999. math.QA/9904164.

[68] David Hestenes. *Space-Time algebra*. Gordon & Breach, New York, 1966.

[69] David Hestenes and Garret Sobczyk. *Clifford algebra to geometric calculus*. Kluwer Academic Publishers, Dordrecht, 1992. 1st. ed 1984, Reprinted with corrections.

[70] David Hestenes and Renatus Ziegler. Projective geometry with Clifford algebra. *Acta Appl. Math.*, 23:25–63, 1991.

[71] David Hilbert. *Die Grundlagen der Geometrie*. B.G. Teubner, Stuttgart, 1968. (first edition from 1899).

[72] Heinz Hopf. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen. *Annals of Mathematics*, 42(1):22–52, 1941.

[73] Adolf Hurwitz. Über die Komposition der quadratischen Formen. *Math. Ann.*, 88:1–25, 1923.

[74] D. L. Johnson. *Presentation of Groups*. Cambridge University Press, Cambridge, 1997. (first published 1990).

[75] Louis H. Kauffman. *Knots and physics*. World Scientific, Singapore, 1991.

[76] Louis H. Kauffman. Virtual knot theory. *European Journal of Combinatorics*, 20:663–690, 1999.

[77] Louis H. Kauffman and David E. Radford. On two proofs for the existence and uniqueness of integrals for finite dimensional Hopf algebras. 1999. math.QA/9911081.

[78] Gregory Maxwell Kelly and Miguel L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[79] Ralf Kerschner. *Effektive Dynamik zusammengesetzter Quantenfelder*. PhD thesis, Universität Tübingen, Tübingen, 1994.

[80] Vladimir V. Kisil. Polynomial sequences of binomial type and path integrals. 1998. math.CO/9809040.

[81] Dirk Kreimer. On the Hopf algebra structure of perturbative quantum field theory. *Adv. Th. Math. Phys.*, 2:303–334, 1998.

[82] Dirk Kreimer. *Knots and Feynman Diagrams*. Cambridge University Press, Cambridge, 2000.

[83] Joseph P. S. Kung and Gian-Carlo Rota. The invariant theory of binary forms. *Bulletin of the American Mathematical Society*, 10(1):27–85, 1984.

[84] Greg Kuperberg. Involutory Hopf algebras and 3-manifold invariants. *International Journal of Mathematics*, 2(1):41–66, 1991.

[85] Greg Kuperberg. Noninvolutory Hopf algebras and 3-manifold invariants. *Duke Mathematical Journal*, 84(1):83–129, July 1996.

[86] Alfred Lotze. Über eine neue Begründung der regressiven Multiplikation extensiver Größen in einem Hauptgebiet n-ter Stufe. *Jahresbericht der DMV*, 57:102–110, 1955.

[87] Pertti Lounesto. *Clifford Algebras and Spinors*. Cambridge University Press, Cambridge, 2001. (first published 1997).

[88] Volodimir Lyubashenko. Modular transformations for tensor categories. *Journal of Pure and Applied Algebra*, 98:279–327, 1995.

[89] Volodimir Lyubashenko. Tangles and Hopf algebras in braided categories. *Journal of Pure and Applied Algebra*, 98:245–278, 1995.

[90] Saunders Mac Lane. *Categories for Working Mathematicians*. Springer-Verlag, New York, 1971.

[91] Shahn Majid. *Foundations of Quantum Group Theory*. Cambridge University Press, Cambridge, 1995.

[92] Waterloo Maple. *Maple V rel. 5.1*. Maplesoft, Univ. of Walterloo, CA. `http://www.maplesoft.com/`.

[93] A. Micali, R. Boudet, and J. Helmstetter, editors. *Clifford Algebras and their Applications in Mathematical Physics*, Dordrecht, 1992. Kluwer Academic Publishers. Montpellier 1989.

[94] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Annals of Mathematics*, 81:211–264, 1965.

[95] Gerard J. Murphy. *C^*-Algebras and Operator Theory*. Academic Press, Boston, 1990.

[96] F. Nill, K. Szlachány, and H.-W. Wiesbrock. Weak Hopf algebras and reducible Jones inclusions of depth 2. – I: From crossed products to Jones towers. 1998. math.QA/98061.

[97] Florian Nill. Axioms for weak bialgebras. 1998. math.QA/9805104.

[98] John J. O'Connor and Edmund F. Robertson. The MacTutor History of Mathematics archive. September 2001. `http://www-history.mcs.st-andrews.ac.uk/history/index.html`.

[99] Zbigniew Oziewicz. Operad of graphs, convolution and Hopf gebra. *Contemporary Mathematics*, pages 1–17. submitted.

[100] Zbigniew Oziewicz. From Grassmann to Clifford. In J.S.R. Chisholm and A.K. Common, editors, *Clifford Algebras and their Applications in Mathematical Physics*, pages 245–256, Dordrecht, 1986. Kluwer. Canterbury, 1985.

[101] Zbigniew Oziewicz. Clifford algebras of multivectors: The minimum polynomials of the tensor product Dirac matrices and the opposite Clifford algebra. In *The Theory of the Electron*, pages 467–486, México, 1997. Universidad Nacional Autónoma de México.

[102] Zbigniew Oziewicz. Clifford Hopf gebra and biuniversal Hopf gebra. *Czechoslovak Journal of Physics*, 47(12):1267–1274, 1997. q-alg/9709016.

[103] Zbigniew Oziewicz. Guest editor's note: Clifford algebras and their applications. *Internation Journal of Theoretical Physics*, 40(1):1–13, 2001.

[104] Zbigniew Oziewicz and Jose de Jesu de Cruz Guzman. Unital and antipodal biconvolution and Hopf gebra. *Miscellanea Algebraicae*, 2001. in press.

[105] Giuseppe Peano. *Calcolo Geometrico; secondo l'Ausdehnungslehre di H. Graßmann, preceduto dalle Operatzioni della Logica Deduttiva*. Fratelli Bocca Editori, Torino, 1888.

[106] Roger Penrose and Wolfgang Rindler. *Spinors and space–time : Two-spinor calculus and relativistic fields*. Cambridge University Press, Cambridge, 1984.

[107] Roger Penrose and Wolfgang Rindler. *Spinors and space–time : Spinor and twistor methods in space-time geometry*. Cambridge University Press, Cambridge, 1986.

[108] Winfried Pfister. Der einzeitige Formalismus in der funktionalen Quantentheorie, 1987.

[109] Winfried Pfister. *Yang-Mills-Dynamik als effektive Theorie von vektoriellen Spinor–Isospinor–Bindungszuständen in einem Preonfeldmodell*. PhD thesis, Universität Tübingen, Tübingen, 1990.

[110] G Pinter. Finite renormalization in the Epstein-Glaser framework and renormalization of the S-matrix of Φ^4-theory. *Ann. Phys. (Leipzig)*, 10:333–363, 2001.

[111] G. Pinter. The Hopf algebra structure of Connes and Kreimer in Epstein-Glaser renormalization. *Lett. Math. Phys.*, 54:227–233, 2002.

[112] Ian Porteous. *Topological Geometry*. Van Nostrand Reinhold, Company, London, 1969.

[113] Ian R. Porteous. *Clifford algebras and the classical groups*. Number I–X,1–29. Cambridge Univ. Press, Cambridge, 2000. Cambridge studies in advanced mathematics ; Vol. 50.

[114] J. Radon. Lineare Scharen orthogonaler Matrizen. *Abh. Math. Sem. Hamburg*, 1(1.14):1–24, 1923.

[115] Marcel Riesz. *Clifford Numbers and Spinors*. Kluwer Academic Publishers, Dordrecht, 1957. (Lecture series *Clifford Numbers and Spinors* given by Professor Marcel Riesz, October 1957 - January 1958, Faximile reprint by Kluwer 1993).

[116] Gian-Carlo Rota. *Indiscrete Thoughts*. Birkhäuser, Boston, 1997.

[117] Gian-Carlo Rota and Stein Joel. Applications of Cayley algebras. *Accad. Naz. Lincei, Rome*, Colloquio:71–97, 1976.

[118] Gian-Carlo Rota and Joel A. Stein. Plethystic algebras and vector symmetric functions. *Proc. Natl. Acad. Sci. USA*, 91:13062–13066, December 1994.

[119] Gian-Carlo Rota and Joel A. Stein. Plethystic Hopf algebras. *Proc. Natl. Acad. Sci. USA*, 91:13057–13061, December 1994.

[120] John Ryan and Wolfgang Sprößig, editors. *Clifford algebras and their Applications in Mathematical Physics – Vol.2: Clifford Analysis*. Birkhäuser, Boston, 2000.

[121] Heinrich Saller. Quantum algebras I: Basic algebraic and topological structures. *Nuovo Cim.*, 108(6):603–630, 1993.

[122] Heinrich Saller. Quantum algebras II: Tangent and gauge structures. *Nuovo Cim.*, 109:255–280, 1994.

[123] Heinrich Saller. Brevier. München, 1996. (unpublished).

[124] Günter Scheja and Uwe Storch. *Lehrbuch der Algebra : unter Einschluß der linearen Algebra*, volume I. Teubner, Stuttgart, 1980.

[125] Günter Scheja and Uwe Storch. *Lehrbuch der Algebra : unter Einschluß der linearen Algebra*, volume II. Teubner, Stuttgart, 1988.

[126] William R. Schmitt. Antipodes and incidence coalgebras. *Journal of Combinatorial Theory*, Series A 4:264–290, 1987.

[127] Gert Schubring, editor. *Hermann Günther Graßmann (1809–1877): Visionary Mathematician, Scientist and Neohumanist Scholar*, Dordrecht, 1996. Kluwer Academic Publishers.

[128] Harald Stumpf and Thomas Borne. *Composite Particle Dynamics in Quantum Field Theory*. Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig, 1994.

[129] Harald Stumpf, Bertfried Fauser, and Winfried Pfister. Composite particle theory in quantum electrodynamics. *Zeitschrift für Naturforschung*, 48(a):765–776, 1993.

[130] Moss E. Sweedler. *Hopf Algebras*. W. A. Benjamin, INC., New York, 1996.

[131] Jose G. Vargas and Douglas G. Torr. Quantum Clifford algebra from classical differential geometry. *J. Math. Phys.*, 2002. in press.

[132] Bartel L. van der Waerden. *A history of algebra : from al-Khwarizmi to Emmy Noether*. Springer-Verlag, Berlin, 1985.

[133] Hassler Whitney. Sphere-Spaces. *Proc. Natl. Acad. Sci. U.S.A.*, 21:464–468, 1935.

[134] Robert Wisbauer. *Grundlagen der Modul- und Ringtheorie*. Verlag Reinhard Fischer, München, 1988.

[135] Edward Witten. Quantum field theory and the Jones polynomial. In C.N. Yang and Gee M.L., editors, *Braid Group, Knot Theory and Statistical Mechanics*, pages 239–329, Singapore, 1989. World Scientific.

[136] Stanislaw Lech Woronowicz. Compact matrix pseudogroups. *Commun. Math. Phys.*, 111:613–665, 1987.

[137] Stanislaw Lech Woronowicz. Differential calculus on compact matrix pseudo groups (quantum groups). *Commun. Math. Phys.*, 122:125–170, 1989.

[138] Francis Wright. *Computing with Maple*. Chapman & Hall/CRC, Boca Raton, 2002.

[139] David N. Yetter. Quantum groups and representations of monoidal categories. *Math. Proceedings of the Cambridge Philosophical Society*, 108:261–290, 1990.

[140] Arno Zaddach. *Graßmanns Algebra in der Geometrie, mit Seitenblicken auf verwandte Strukturen*. BI Wissenschaftsverlag, Mannheim, 1994.

[141] Renatus Ziegler. *Die Geschichte der klassischen Mechanik im 19. Jahrhundert*. Franz Steiner Verlag Wiesbaden GMBH, Stuttgart, 1985.