Abstract

The character ring $\text{Char-GL}$ of covariant irreducible tensor representations of the general linear group admits a Hopf algebra structure isomorphic to the Hopf algebra $\text{Symm-}\Lambda$ of symmetric functions. Here we study the character rings $\text{Char-O}$ and $\text{Char-Sp}$ of the orthogonal and symplectic subgroups of the general linear group within the same framework of symmetric functions. We show that $\text{Char-O}$ and $\text{Char-Sp}$ also admit natural Hopf algebra structures that are isomorphic to that of $\text{Char-GL}$, and hence to $\text{Symm-}\Lambda$. The isomorphisms are determined explicitly, along with the specification of standard bases for $\text{Char-O}$ and $\text{Char-Sp}$ analogous to those used for $\text{Symm-}\Lambda$. A major structural change arising from the adoption of these bases is the introduction of new orthogonal and symplectic Schur-Hall scalar products. Significantly, the adjoint with respect to multiplication no longer coincides, as it does in the $\text{Char-GL}$ case, with a Foulkes derivative or skew operation. The adjoint and Foulkes derivative now require separate definitions, and their properties are explored here in the orthogonal and symplectic cases. Moreover, the Hopf algebras $\text{Char-O}$ and $\text{Char-Sp}$ are not self-dual. The dual Hopf algebras $\text{Char-O}^*$ and $\text{Char-Sp}^*$ are identified. Finally, the Hopf algebra of the universal rational character ring $\text{Char-GLrat}$ of mixed irreducible tensor representations of the general linear group is introduced and its structure maps identified.

Keywords: Orthogonal group, symplectic group, general linear group, irreducible characters, symmetric functions, representation rings, Hopf algebra, group characters, universal rational characters

1 Introduction

1.1 Motivation

It is hardly possible to overestimate the importance of group representation theory and the associated calculus of group characters. It plays a role in many areas of physics, chemistry, biology...
and not least in pure mathematics. For that reason, new techniques which deal with group characters in a unified and structural way are not only of interest in their own right, but also may be of great help in more applied work.

A common problem involving the application of group representation theory is that of determining an underlying symmetry group whose irreducible representations accommodate elementary particle or nucleon states. In this context tensor products govern such things as interactions, scattering and decay processes. In addition symmetry breaking, whereby the potential symmetries of some idealised system are more realistically limited to some subset of the original symmetries, manifests itself group theoretically by way of restriction from group to subgroup and the corresponding branching, or reduction of representations. Traditional homelands in physics for group theory-powered insights have included the multiplet organization of elementary particles, Wigner’s nuclear multiplet theory, the nuclear interacting boson model, atomic and nuclear shell theory, as well as the building of grand unified theories.

In all these cases, while the detailed construction of explicit group representations might be helpful, it is their characters that play the key role. In the present paper we study group representations via the Hopf algebraic structure of these characters. It is the products and coproducts of these Hopf algebras that determine the decompositions of tensor products and group-subgroup branching rules that are required in physical applications, while the characters themselves, which as we shall see are all expressible within the framework of the ring $\Lambda$ of symmetric functions, that specify the physical states themselves.

It was already observed in earlier work [11, 13] that Hopf algebra techniques allowed symmetric function methods to be organized and generalized in an elegant way. The approach was developed in part by applying methods borrowed from quantum field theory [7, 5], in a simplified group theoretical setting. In group theory terms, this earlier symmetric function work concerns the characters of the general linear group. In the present paper, we pursue these investigations by turning to the classical subgroups of the general linear group. We show how the character rings of the orthogonal and symplectic groups admit natural Hopf algebraic structures. We obtain these Hopf algebras as isomorphic images of the Hopf algebra of the character ring of the general linear group, which is in turn isomorphic to the Hopf algebra of symmetric functions. The isomorphy is defined by the underlying branching, which establishes an isomorphism between the module of characters of the general linear group, and those of its classical subgroups. This module map induces a map of Hopf algebras, as we are going to show.

Despite their isomorphism as Hopf algebras, the different character Hopf algebras encode different information. This stems partly from the fact that we are interested in canonical bases, which differ for different character modules. The prime example concerns the Schur functions which furnish irreducible characters of the general linear group GL. If we branch from GL say to the orthogonal group O, or the symplectic group Sp, the Schur functions are no longer the irreducible characters, and they lose, in part, their important and singular meaning. The orthogonality of irreducible GL characters, corresponding to irreducible representations with highest weight specified by integer partitions, $\lambda$, is expressed formally by means of the Schur-Hall scalar product with respect to which the Schur functions $s_\lambda$ are orthonormal,

$$\langle s_\lambda \mid s_\mu \rangle = \delta_{\lambda,\mu}. \quad (1)$$
The decomposition of irreducible representations of $\text{GL}$ on restriction to the orthogonal or symplectic subgroups, $\text{O}$ or $\text{Sp}$, involves a branching rule that is determined by expressing suitably restricted irreducible characters of $\text{GL}$ in terms of irreducible orthogonal or symplectic group characters. These characters will be called Schur functions of orthogonal or symplectic type. Since orthogonal and symplectic groups are completely reducible, we can find a basis of such irreducible characters. It is hence a group-theoretical necessity to introduce, on these character Hopf algebras, new Schur-Hall scalar products which express the fact that Schur functions of orthogonal or symplectic type, $o_\lambda$ or $sp_\lambda$, respectively, are mutually orthonormal (Schur’s lemma):

$$\langle o_\lambda | o_\mu \rangle_2 = \delta_{\lambda,\mu} \quad \text{and} \quad \langle sp_\lambda | sp_\mu \rangle_{11} = \delta_{\lambda,\mu}.$$  \hspace{1cm} (2)

The indexing stems from the plethystic origin of these particular branchings (see below). These scalar products and the associated orthogonal bases are the new structural elements which distinguish the otherwise isomorphic Hopf algebras.

The general case of symmetric function branchings was discussed in [13]. There we considered module isomorphisms between the module of characters of a group $G$ and the module of characters of a subgroup $H$. Specifically, an algebraic subgroup $H_\pi$ of $\text{GL}$ was taken, consisting of matrix transformations fixing an arbitrary tensor of symmetry type $\pi$ – the orthogonal and symplectic cases correspond to the weight two symmetric and antisymmetric cases, $\pi = (2)$ and $\pi = (1,1)$ respectively, of nonsingular bilinear forms. However, generically, the symmetric function bases obtained by branchings with respect to higher rank invariants are no longer irreducible, but only indecomposable at best. For this reason we study, in a first attempt, the orthogonal and symplectic cases.

Even these classical cases reveal some novel features when treated in this formal setting. We need to introduce new classes of Schur functions, as described above, as is well-understood classically and was used at least implicitly already by Weyl. Complete and elementary symmetric functions now have different expansions in terms of irreducible orthogonal and symplectic characters; also it turns out that power sums pick up an extra additive term. More significantly, we need to separate the notion of multiplicative adjoint (which we denote by $s_\lambda^\dagger$), which leads to skew Schur functions in the $\text{GL}$ case, from that of the Foulkes derivative, which we denote by $s_\lambda^\perp$. This stems from the fact that the adjoint of multiplication depends on the Schur-Hall scalar product adopted, and that the branched Hopf algebras are no longer self dual.

This work extends and elaborates on the material presented in the conference paper [20] with the addition of many proofs, and thereby establishes the necessary tools to deal, for example, with vertex operator algebras of orthogonal and symplectic type, as described elsewhere [14], see also for example [2].

A further extension presented here covers rational characters of mixed tensor irreducible representations of the general linear group. In order to exploit Hopf algebra methods systematically in this context, and to make more rigorous previous discussions of products and branchings of mixed tensor representations [1, 18], it is necessary to extend the underlying ring of symmetric functions from $\Lambda$ to $\Lambda \otimes \overline{\Lambda}$ and to define, following Koike [22], corresponding universal rational characters.
1.2 Organisation

The organisation of the paper is as follows. Some facts about the symmetric function Hopf algebra $\text{Symm-}\Lambda$ [11] are provided in Section 2, with an emphasis on its Schur function basis. This section also includes the definitions of certain infinite series of Schur functions. These are used in Section 3 to define the universal irreducible characters of the classical orthogonal and symplectic groups [25, 18, 23] that are the main focus of our study.

Thanks to their definition by way of certain branchings from the general linear group, whose universal irreducible characters are nothing other than Schur functions, the corresponding orthogonal and symplectic character rings are actually Hopf algebras isomorphic to the universal Hopf algebra of symmetric functions $\text{Symm-}\Lambda$ [13]. A complete directory is given in Table 1 of Section 3 of the action of the structure maps of these Hopf algebras as first described in abbreviated form in [20]. Full proofs of all the structure map identities are provided here in Section 4. This is followed in Section 5 by a discussion of the different realizations of the power sum, complete and elementary symmetric functions in the orthogonal and symplectic cases, noting where differences due to the underlying groups occur.

The new orthogonal and symplectic scalar products are introduced in Section 6, in which we discuss the adjoint operation and the Foulkes derivative, and provide the correct Hopf-theoretical definition of the latter, which allows applications to generic branchings. Finally, in this section, the dual Hopf algebras $\text{Char-O}^*$ and $\text{Char-SP}^*$ are identified along with their structure maps. These dual Hopf algebras share the same structure maps as the original Hopf algebras but are isomorphic to $\text{Symm-}\Lambda$ extended so as to include infinite series of Schur functions.

The universal rational characters associated with mixed tensor irreducible representations of $\text{GL}$ are introduced in Section 7. They form the basis of a ring $\Lambda \otimes \overline{\Lambda}$ and an associated new rational character Hopf algebra, $\text{Char-GLrat}$. Explicit expressions are derived for the corresponding products and coproducts, and the remaining Hopf algebra structure maps are also identified explicitly.

2 Symmetric functions and their Hopf algebra

2.1 The ring of symmetric functions $\Lambda$

We use the standard notation of Macdonald’s book [27]. Symmetric functions are conveniently indexed by integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The $\lambda_i \in \mathbb{N}$ are the parts of the partition, $|\lambda| = \sum_{i=1}^\ell \lambda_i$ is the weight of the partition, while $\ell(\lambda) = \ell$ is its length. Such a partition is often written without the trailing zeros. In exponent form $\lambda = (\ldots, k^{m_k}, \ldots, 2^{m_2}, 1^{m_1})$ where $m_k \geq 0$ is the number of parts of $\lambda$ that are equal to $k$ for $k = 1, 2, \ldots$. With this notation it is convenient to introduce $z_\lambda = \prod_{k \geq 1} k^{m_k} m_k!$

Corresponding to each partition $\lambda$ there exists a Ferrers or Young diagram $F^\lambda$. This consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths $\lambda_i$ for $i = 1, 2, \ldots, \ell(\lambda)$. The column lengths of $F^\lambda$ specify the parts $\lambda'_j$ for $j = 1, 2, \ldots, \ell(\lambda')$ of the partition $\lambda'$ that is conjugate to $\lambda$. If $F^\lambda$ has $r$ boxes on the main diagonal, with arm lengths $\lambda_k - k = a_k$ and leg lengths $\lambda'_k - k = b_k$
for \( k = 1, 2, \ldots, r \), then \( \lambda \) is said to have rank \( r(\lambda) = r \) and in Frobenius notation \( \lambda = (a_1 a_2 \cdots a_r) \) and \( \lambda' = (b_1 b_2 \cdots b_r) \) with \( a_1 > a_2 > \cdots > a_r \geq 0 \) and \( b_1 > b_2 > \cdots > b_r \geq 0 \). Schematically, we have

\[
\begin{array}{|c|c|c|c|c|}
\hline
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\hline
\hline
\end{array}
= \begin{array}{|c|c|c|c|c|}
\hline
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\
\hline
\end{array}
= \begin{array}{|c|c|c|c|c|}
\hline
a_1 & a_2 & a_3 & a_4 & a_5 \\
\hline
\end{array}
= \begin{array}{|c|c|c|c|c|}
\hline
b_1 & b_2 & b_3 & b_4 & b_5 \\
\hline
\end{array}
\]

By way of an example we have

\[
(7, 4, 4, 2, 0, \ldots) = (7, 4^2, 2) = \left(\begin{array}{ccc}
6 & 2 & 1 \\
3 & 2 & 0
\end{array}\right)
\]

and its conjugate

\[
(4, 4, 3, 1, 1, 1, 0 \ldots) = (4^2, 3^2, 1^3) = \left(\begin{array}{ccc}
3 & 2 & 0 \\
6 & 2 & 1
\end{array}\right).\]

Partitions are used to specify a number of objects of interest in the present work. Amongst these are the Schur functions \( s_\lambda \). These form an orthonormal \( \mathbb{Z} \)-basis for the ring \( \Lambda \) of symmetric functions. To be more precise, let \( \mathbb{Z}[x_1, \ldots, x_N] \) be the polynomial ring, or the ring of formal power series, in \( N \) commuting variables \( x_1, \ldots, x_N \). The symmetric group \( S_N \) acting on \( N \) letters acts on this ring by permuting the variables. For \( \pi \in S_N \) and \( f \in \mathbb{Z}[x_1, \ldots, x_N] \) we have

\[
\pi f(x_1, \ldots, x_N) = f(x_{\pi(1)}, \ldots, x_{\pi(N)}). \quad (3)
\]

We are interested in the subring of functions invariant under this action, \( \pi f = f \), that is to say the ring of symmetric polynomials in \( N \) variables:

\[
\Lambda_N[x_1, \ldots, x_N] = \mathbb{Z}[x_1, \ldots, x_N]^{S_N}. \quad (4)
\]

This ring may be graded by the degree of the polynomials, so that

\[
\Lambda_N[x_1, \ldots, x_N] = \bigoplus_n \Lambda_N^{(n)}[x_1, \ldots, x_N], \quad (5)
\]

where \( \Lambda_N^{(n)}[x_1, \ldots, x_N] \) consists of homogenous symmetric polynomials in \( x_1, \ldots, x_N \) of total degree \( n \).

In order to work with an arbitrary number of variables, following Macdonald [27], we define the ring of symmetric functions \( \Lambda = \lim_{N \to \infty} \Lambda_N \) in its stable limit \( (N \to \infty) \) where \( \Lambda_N = \Lambda_M[x_1, \ldots, x_N, 0, \ldots, 0] \) for all \( M \geq N \). This ring of symmetric functions inherits the grading \( \Lambda = \bigoplus_n \Lambda^{(n)} \), with \( \Lambda^{(n)} \) consisting of homogeneous symmetric polynomials of degree \( n \).

A \( \mathbb{Z} \) basis of \( \Lambda^{(n)} \) is provided by the monomial symmetric functions \( m_\lambda \) where \( \lambda \) is any partition of \( n \). There exist further (integral and rational) bases for \( \Lambda^{(n)} \) that are indexed by the
generating functions take the form
\[ h_n = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_t}, \quad e_n = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_t}, \quad p_n = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_t}, \] (6)
where the one part functions are defined for all \( n \in \mathbb{N} \) by
\[ h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad p_n = \sum_i x_i^n. \] (7)
With the convention \( h_0 = e_0 = p_0 = 1 \) and \( h_{-n} = e_{-n} = p_{-n} = 0 \) for positive \( n \), their generating functions take the form
\[ H_t = \prod_{i \geq 1} \frac{1}{(1 - x_i t)}, \quad E_t = \prod_{i \geq 1} (1 + x_i t) = \sum_{n \geq 0} e_n t^n, \quad t \frac{d}{dt} \log H_t = \sum_{n \geq 1} p_n t^n. \] (8)

The most important non-multiplicative basis of \( \Lambda^{(n)} \) is provided by the Schur functions \( s_\lambda \) with \( \lambda \) running over all partitions of \( n \). For a finite number of variables the Schur function \( s_\lambda(x_1, \ldots, x_N) \) may be defined as a ratio of alternants. It is a homogeneous symmetric polynomial of total degree \( n \), and is stable in the sense that \( s_\lambda(x_1, \ldots, x_N, 0, \ldots, 0) = s_\lambda(x_1, \ldots, x_N) \) regardless of how many 0’s are appended to the list of variables. Taking the limit as \( N \to \infty \) of \( s_\lambda(x_1, \ldots, x_N) \) serves to define the required \( s_\lambda \in \Lambda^{(n)} \) [27].

Varying \( \lambda \) over all partitions, the Schur functions \( s_\lambda \) provide a \( \mathbb{Z} \)-basis of \( \Lambda \). We can go further. There exists a bilinear form on \( \Lambda \), the Schur-Hall scalar product \( \langle \cdot | \cdot \rangle \). With respect to this scalar product, the Schur functions form an orthonormal basis of \( \Lambda \). In fact we have:
\[ \langle s_\lambda | s_\mu \rangle = \delta_{\lambda,\mu}, \quad \langle p_\lambda | p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \quad \langle m_\lambda | h_\mu \rangle = \delta_{\lambda,\mu}, \quad \langle f_\lambda | e_\mu \rangle = \delta_{\lambda,\mu}. \] (9)

These relations serve to define the ‘monomial’ symmetric functions \( m_\lambda \), and the so-called ‘forgotten’ symmetric functions \( f_\lambda \) (for details see [27]).

In what follows we make use of various notation for Schur functions, including for example \( s_\lambda(x_1, \ldots, x_N), s_\lambda(x), s_\lambda(x, y) \) or \( s_\lambda \), depending on whether or not it is necessary to be explicit about the number of variables or the sets of variables under consideration. Here, a single symbol \( x \) may often stand for an alphabet, \( x_1, x_2, \ldots \), finite or otherwise, while a pair \( x, y \) signifies a pair of such alphabets \( x_1, x_2, \ldots, y_1, y_2, \ldots \).

### 2.2 The Hopf algebra \textbf{Symm-}\( \Lambda \)

The graded ring of symmetric functions \( \Lambda \) spanned by the Schur functions \( s_\lambda \) affords a graded self-dual, bicommutative Hopf algebra, which we denote by \textbf{Symm-}\( \Lambda \), as can be seen once we have identified the appropriate product, coproduct, unit, counit, antipode and self-duality condition. This can be done as follows.
The **outer product** of Schur functions is given in prefix form, infix dot product form, first without and then with variables, and finally in explicit form:

\[
m(s_\mu \otimes s_\nu) = s_\mu \cdot s_\nu ,
\]

\[
m(s_\mu(x) \otimes s_\nu(y)) = s_\mu(x) \cdot s_\nu(x) = \sum_{\lambda} C^\lambda_{\mu,\nu} s_\lambda(x) .
\]

(10)

In some situations the \( \cdot \) is omitted in favour of simple juxtaposition. Here, and elsewhere if not otherwise specified, tensor products are over \( \mathbb{Z} \) (or \( \mathbb{Q} \) if power sums are involved).

The **outer coproduct** map is denoted by \( \Delta \), and we use the variable or the tensor product notation interchangeably, to give the coproduct in prefix form, Sweedler index form [35] and skew product forms, first without and then with variables, and finally in explicit form:

\[
\Delta(s_\lambda) = s_{\lambda(1)} \otimes s_{\lambda(2)} = \sum_{\nu} s_{\lambda/\nu} \otimes s_\nu = \sum_{\mu} s_\mu \otimes s_{\lambda/\mu} ,
\]

\[
\Delta(s_\lambda(x,y)) = s_{\lambda(x,y)} = s_{\lambda(1)}(x)s_{\lambda(2)}(y) = \sum_{\mu,\nu} C^\lambda_{\mu,\nu} s_\mu(x)s_\nu(y) .
\]

(11)

The coefficient \( C^\lambda_{\mu,\nu} \) of the multiplication map \( m \) in the Schur function basis, and the structure constant \( C^\lambda_{\mu,\nu} \) of the coproduct in the same basis turn out to be identical. This equality of coefficients is a consequence of the self-duality condition

\[
\langle s_\lambda | m(s_\mu \otimes s_\nu) \rangle = \langle \Delta(s_\lambda) | s_\mu \otimes s_\nu \rangle .
\]

(12)

In fact, although logically distinct, they are both equal to the famous Littlewood-Richardson coefficient \( c^\lambda_{\mu,\nu} \), which may be evaluated combinatorially using the Littlewood-Richardson rule [25, 27]. Thus we have

\[
C^\lambda_{\mu,\nu} = c^\lambda_{\mu,\nu} = C^\lambda_{\nu,\mu} .
\]

(13)

This follows from the well know fact that the dot and skew products of Schur functions are adjoint with respect to the Schur-Hall scalar product, that is to say [27]

\[
\langle s_\lambda | s_\mu \cdot s_\nu \rangle = c^\lambda_{\mu,\nu} = \langle s_{\lambda/\nu} | s_\mu \rangle .
\]

(14)

Here, in the Schur function basis the operations of outer multiplication and that of skewing are both defined in terms of Littlewood-Richardson coefficients by

\[
s_\mu \cdot s_\nu = \sum_{\lambda} c^\lambda_{\mu,\nu} s_\lambda \quad \text{and} \quad s^\perp_\mu(s_\lambda) = s_{\lambda/\mu} = \sum_{\nu} c^\lambda_{\mu,\nu} s_\nu .
\]

(15)

The notation \( s^\perp_\mu(s_\lambda) \) has been introduced to emphasise the fact that the ring of symmetric functions has a module structure under \( ^\perp \):

\[
f^\perp(g^\perp(h)) = (gf)^\perp(h) \quad \text{or equivalently} \quad (h/g)f = h/(gf) .
\]

(16)
The unit map $\eta$, counit map $\epsilon$, and antipode $S$, are defined by:

$$\eta : 1 \rightarrow s_0, \quad \epsilon : s_\lambda \rightarrow \delta_{\lambda,0}, \quad S : s_\lambda \rightarrow (-1)^{|\lambda|} s'_\lambda.$$ \hfill (17)

It is important to note that the following antipode identity in the Hopf algebra $\text{Symm-}\Lambda$:

$$m (I \otimes S) \Delta (s_\lambda) = \eta \epsilon (s_\lambda)$$ \hfill (18)

yields the result [11]

$$\sum_\nu (-1)^{|\nu|} s_{\lambda/\nu} \cdot s_{\nu'} = \delta_{\lambda,0} s_0,$$ \hfill (19)

since

$$m(I \otimes S) \Delta (s_\lambda) = m(I \otimes S) (\sum_\nu s_{\lambda/\nu} \otimes s_\nu)$$
$$= m (\sum_\nu s_{\lambda/\nu} \otimes (-1)^{|\nu|} s_{\nu'}) = \sum_\nu (-1)^{|\nu|} s_{\lambda/\nu} \cdot s_{\nu'}$$

and

$$\eta \epsilon (s_\lambda) = \eta(\delta_{\lambda,0}) = \delta_{\lambda,0} s_0.$$ 

Returning to the bases provided by $h_\lambda$, $e_\lambda$ and $p_\lambda$ in (6), these bases are so-called multiplicative, because the outer product is just the (unordered) concatenation product. Using self-duality, this means that the coproduct is just deconcatenation of these products, together with the action:

$$\Delta (h_n) = \sum_{r=0}^n h_{n-r} \otimes h_r, \quad \Delta (e_n) = \sum_{r=0}^n e_{n-r} \otimes e_r, \quad \Delta (p_n) = p_n \otimes 1 + 1 \otimes p_n.$$ \hfill (20)

These results all follow immediately from the definitions of (8). The first two of these show that one part complete and elementary symmetric functions are divided powers. The third shows that the one part power sum symmetric functions are the primitive elements of the Hopf algebra $\text{Symm-}\Lambda$.

2.3 Schur function series

To describe characters of the orthogonal and symplectic groups effectively, Littlewood [25] introduced a set of infinite series of Schur functions which we are frequently going to use; consult also [18, 4]. It is convenient to extend our ring $\Lambda$ to $\Lambda[[t]]$, where $t$ is a formal parameter, and our Hopf algebra to $\text{Symm-}\Lambda[[t]]$ itself extended so as to encompass infinite series of Schur
Some of these Schur function series read

\[ A_t = \sum_{\alpha \in \mathcal{A}} (-1)^{|\alpha|/2} t^{\alpha} \{\alpha\} \]
\[ B_t = \sum_{\beta \in \mathcal{B}} t^{|\beta|} \{\beta\} \]
\[ C_t = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|/2} t^{\gamma} \{\gamma\} \]
\[ D_t = \sum_{\delta \in \mathcal{D}} t^{\delta} \{\delta\} \]
\[ E_t = \sum_{\epsilon \in \mathcal{E}} (-1)^{|\epsilon|+(\ell(\epsilon))/2} t^{\epsilon} \{\epsilon\} \]
\[ F_t = \sum_{\zeta \in \mathcal{F}} t^{\zeta} \{\zeta\} \]
\[ G_t = \sum_{\tau \in \mathcal{G}} (-1)^{|\tau|-(\ell(\tau))/2} t^{\tau} \{\tau\} \]
\[ H_t = \sum_{\xi \in \mathcal{H}} (-1)^{|\xi|} t^{\xi} \{\xi\} \]
\[ L_t = \sum_{m \geq 0} (1 + t)^m \{1^m\} \]
\[ M_t = \sum_{m \geq 0} t^m \{m\} \]
\[ P_t = \sum_{m \geq 0} (1 + t)^m \{m\} \]
\[ Q_t = \sum_{m \geq 0} t^m \{1^m\} \] (21)

where \( m \) is summed over all non-negative integers, while \( \mathcal{P} \) is the set of all partitions, \( \mathcal{D} = 2\mathcal{P} \) is the set of partitions all of whose parts are even, and \( \mathcal{B} \) is the set of partitions that are conjugate to those of \( \mathcal{D} \). To define \( \mathcal{A} \), \( \mathcal{C} \) and \( \mathcal{E} \), it is convenient in Frobenius notation to let

\[ \mathcal{P}_n = \left\{ \left( \begin{array}{c} a_1 \ a_2 \ \cdots \ a_r \\ b_1 \ b_2 \ \cdots \ b_r \end{array} \right) \in \mathcal{P} \left| \begin{array}{l} a_k - b_k = n \quad \text{for all} \quad r = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots, r \end{array} \right. \right\} \] (22)

for all integers \( n \). With this notation \( \mathcal{A} = \mathcal{P}_{-1}, \mathcal{C} = \mathcal{P}_1 \) and \( \mathcal{E} = \mathcal{P}_0 \). Thus the partitions in \( \mathcal{C} \) are the conjugates of those in \( \mathcal{A} \), while \( \mathcal{E} \) is the set of all self-conjugate partitions. It should be pointed out that each of the Schur function series of (21) includes the term \( \{0\} = s_0 = 1 \) since \( \{1^0\} = \{0\} \) and the sets \( \mathcal{P}, 2\mathcal{P} \) and \( \mathcal{P}_n \), for all integers \( n \), contain the zero partition \( \{0\} \), from which \( |\{0\}| = \ell(0) = r(0) = 0 = (0)^r = (0) \).

The generating functions which serve to define these series take the form:

\[ A_t := \prod_{i<j} (1 - t^2 x_i x_j) \]
\[ B_t := \prod_{i<j} (1 - t^2 x_i x_j)^{-1} \]
\[ C_t := \prod_{i<j} (1 - t^2 x_i x_j) \]
\[ D_t := \prod_{i<j} (1 - t^2 x_i x_j)^{-1} \]
\[ E_t := \prod_k (1 - t x_k) \prod_{i<j} (1 - t^2 x_i x_j) \]
\[ F_t := \prod_k (1 - t x_k)^{-1} \prod_{i<j} (1 - t^2 x_i x_j)^{-1} \]
\[ G_t := \prod_k (1 + t x_k) \prod_{i<j} (1 - t^2 x_i x_j) \]
\[ H_t := \prod_k (1 + t x_k)^{-1} \prod_{i<j} (1 - t^2 x_i x_j)^{-1} \]
\[ L_t := \prod_k (1 - t x_k) \]
\[ M_t := \prod_k (1 - t x_k)^{-1} \]
\[ P_t := \prod_k (1 + t x_k)^{-1} \]
\[ Q_t := \prod_k (1 + t x_k) \] (23)
As can be seen there is some redundancy here because \( P_t = M_{-t}, Q_t = L_{-t}, G_t = E_{-t} \) and \( H_t = F_{-t} \), however we keep all 12 series \( Z_t \) because in what follows we often denote \( Z_1 \) by \( Z \).

On the other hand we sometimes need to display the arguments \( x = (x_1, x_2, \ldots) \) of the above series by adopting the more explicit notation \( Z_t(x) \).

A major feature of the above list of Schur function series is that they come, as can be seen from their generating functions, in mutually inverse pairs:

\[
A_t B_t = C_t D_t = E_t F_t = G_t H_t = L_t M_t = P_t Q_t = 1. \tag{24}
\]

Moreover

\[
L_t A_t = P_t C_t = E_t, \quad M_t B_t = Q_t D_t = F_t, \quad M_t C_t = Q_t A_t = G_t, \quad L_t D_t = P_t B_t = H_t. \tag{25}
\]

The above generating functions make it particularly easy to establish the following:

**Proposition 2.1:** Let \( Z_t \) be any one of the Schur function series (23), then

\[
\Delta(Z_t) = (Z_t \otimes Z_t) \Delta'(Z_t) \tag{26}
\]

where the cut coproducts are given by

\[
\Delta'(Z_t) = \begin{cases}
\sum_{\sigma \in P} (-t^2)^{|\sigma|} \{\sigma\} \otimes \{\sigma'\} & \text{for } Z_t = A_t, C_t, E_t, G_t; \\
\sum_{\sigma \in P} t^2|\sigma| \{\sigma\} \otimes \{\sigma\} & \text{for } Z_t = B_t, D_t, F_t, H_t; \\
1 & \text{for } Z_t = L_t, M_t, P_t, Q_t.
\end{cases} \tag{27}
\]

**Proof:** Let \( z = (z_1, z_2, \ldots) = (x, y) = (x_1, x_2, \ldots, y_1, y_2, \ldots) \) and note that \( \Delta(Z_t(z)) = Z_t(x, y) = Z_t(x)Z_t(y)Z_t'(x, y) \) where the generating functions immediately imply that for \( Z_t = A_t, C_t, E_t, G_t \) we have \( Z_t'(x, y) = \prod_{i,j}(1 - t^2 x_i y_j) \), and for \( Z_t = B_t, D_t, F_t, H_t \) we have \( Z_t'(x, y) = \prod_{i,j}(1 - t^2 x_i y_j)^{-1} \), whilst for \( Z_t = L_t, M_t, P_t, Q_t \) we have \( Z_t'(x, y) = 1 \). It only remains to use the well known Cauchy identity [27]

\[
K_t(x, y) := \prod_{i,j}(1 - t^2 x_i y_j)^{-1} = \sum_{\sigma \in P} t^2|\sigma| \{\sigma\} \otimes \{\sigma\} \tag{28}
\]

and its dual

\[
J_t(x, y) := \prod_{i,j}(1 - t^2 x_i y_j) = \sum_{\sigma \in P} (-t^2)^{|\sigma|} \{\sigma\} \otimes \{\sigma'\}. \tag{29}
\]
It might be added here that
\[
\prod_{i,j} (1 - t^2 x_i x_j)^{-1} = B_t(x) \quad D_t(x) = \sum_{\sigma \in \mathcal{P}} t^{2|\sigma|} s_{\sigma}(x) s_{\sigma}(x) = \sum_{\sigma \in \mathcal{P}} t^{2|\sigma|} \{ \sigma \cdot \sigma \}(x)
\] (30)
and
\[
\prod_{i,j} (1 - t^2 x_i x_j) = A_t(x) \quad C_t(x) = \sum_{\sigma \in \mathcal{P}} (-t^2)^{|\sigma|} s_{\sigma}(x) s_{\sigma'}(x) = \sum_{\sigma \in \mathcal{P}} (-t^2)^{|\sigma|} \{ \sigma \cdot \sigma' \}(x). \quad (31)
\]

We are now in a position to exploit these series and their associated identities in the specification of characters of the classical groups, in particular what are known as their universal characters, and to study the detailed properties of the Hopf algebras of their character rings, initiated in [13] and [20].

3 The Hopf algebras of the character rings of classical groups

3.1 Universal characters of covariant tensor representations

The groups, \( G \), under consideration here are the general linear group \( \text{GL} \), the orthogonal group \( O \) and the symplectic group \( \text{Sp} \). If the classical groups \( \text{GL} \), \( O \) and \( \text{Sp} \) act by way of linear transformations in a space \( V \) of dimension \( N \), then they are denoted by \( \text{GL}(N) \), \( O(N) \) and \( \text{Sp}(N) \), respectively. Initially we confine attention to their finite-dimensional irreducible covariant tensor representations \( V^\lambda_G \). Each of these is specified by their highest weight \( \lambda \), which in each case is a partition. The corresponding character is denoted by \( \text{ch} V^\lambda_G \). These characters may be expressed by means of Weyl’s character formula [37] in terms of the eigenvalues \((x_1, \ldots, x_N)\) of each group element \( g \in G \) realised as a matrix \( M \in \text{End}(V) \) of linear transformations in \( V \).

It is well known that in the case of \( \text{GL}(N) \) we have
\[
\text{ch} V^\lambda_{\text{GL}(N)} = \{ \lambda \}(x_1, \ldots, x_N) = s_\lambda(x_1, \ldots, x_N), \quad (32)
\]
where the central symbol accords with the notation of Littlewood [25]. This character shares the same stable \( N \to \infty \) limit as Schur functions, and in this limit we define the \textit{universal character} [23, 19]
\[
\text{ch} V^\lambda_{\text{GL}} = \{ \lambda \} = s_\lambda. \quad (33)
\]

The orthogonal and symplectic groups leave invariant a symmetric second rank tensor \( g_{ij} = g_{ji} \) and an antisymmetric second rank tensor \( f_{ij} = -f_{ji} \), respectively. It is necessary to distinguish between the even and odd cases: \( N = 2K \) and \( N = 2K + 1 \) with \( K \in \mathbb{N} \). The groups \( O(2K) \), \( O(2K+1) \) and \( \text{Sp}(2K) \) are all reductive Lie groups whose finite-dimensional representations are fully reducible. On the other hand \( \text{Sp}(2K+1) \), an odd-dimensional symplectic group, is not reductive. This is a consequence of the fact that its invariant bilinear form is singular. It can be realised as an affine extension of \( \text{Sp}(2K) \times \text{GL}(1) \), that is the semi-direct product of these.
groups with a set of translations as explained by Proctor [31]. As a result its finite-dimensional representations are not necessarily fully reducible. Indeed its defining representation, $V$, of dimension $2K + 1$ is indecomposable but contains two irreducible constituents of dimensions $2K$ and $1$. More generally, Proctor has established that the representations $V_{Sp(2K+1)}^\lambda$ are reducible but indecomposable for $\lambda \neq 0$.

Despite these issues associated with the evenness and oddness of $N$, there still exists a stable $N \to \infty$ limit and associated universal characters [23, 19] denoted here by $chV_O^\lambda$ and $chV_{Sp}^\lambda$. The Schur function series that we have introduced enable us to write down Schur function expressions for these universal characters of $O$ and $Sp$ in the form [25, 18]

$$chV_O^\lambda = o_\lambda = [\lambda] = \{\lambda/C\} = s_{\lambda/C} = \sum_{\gamma \in \mathcal{C}} (-1)^{|\gamma|/2} s_{\lambda/\gamma},$$

$$chV_{Sp}^\lambda = sp_\lambda = \langle \lambda \rangle = \{\lambda/A\} = s_{\lambda/A} = \sum_{\alpha \in \mathcal{A}} (-1)^{|\alpha|/2} s_{\lambda/\alpha}.$$  

These relations are the inverse of the branching rules for the restriction from $GL$ to its subgroups $O$ and $Sp$:

$$chV_{GL}^\lambda = \{\lambda\} = [\lambda/D] = \sum_{\delta \in \mathcal{D}} [\lambda/\delta] = \sum_{\delta \in \mathcal{D}, \zeta \in \mathcal{P}} c_{\delta,\zeta}^\lambda chV_O^\zeta,$$

$$chV_{GL}^\lambda = \{\lambda\} = \langle \lambda/B \rangle = \sum_{\beta \in \mathcal{D}} \langle \lambda/\beta \rangle = \sum_{\beta \in \mathcal{D}, \zeta \in \mathcal{P}} c_{\beta,\zeta}^\lambda chV_{Sp}^\zeta.$$  

That the above pairs of relations are mutually inverse is a simple consequence of the identities $AB = CD = 1$.

In describing the Hopf algebras of the character rings of the groups $GL$, $O$ and $Sp$ we deal only with the universal characters, their restriction to the finite $N$ case necessitates the use of modification rules if the relevant partitions are of too great a length. Further details may be found elsewhere, for example [28, 17, 4, 23].

The Hopf algebra of symmetric functions, $\text{Symm}-\Lambda$, is the universal, graded, bicommutative, biassociative self-dual Hopf algebra. Its properties have been spelt out in the Schur function basis in Section 2.2. Having identified in Section 3.1 the universal characters of the classical groups and expressed them in terms of Schur functions, the Hopf algebras of their universal character rings may be found as isomorphic copies of $\text{Symm}-\Lambda$. Despite the fact that the structure maps acting on the character ring Hopf algebra $\text{Char-GL}$, $\text{Char-O}$ and $\text{Char-Sp}$ are isomorphic to those of $\text{Symm}-\Lambda$, they take different explicit forms in the different canonical bases. These bases are distinguished by the use of different Littlewood parentheses, $\{\lambda\}$, $[\lambda]$ and $\langle \lambda \rangle$ together with their particular Schur-Hall scalar products with respect to which the bases are orthogonal.

3.2 The general linear case

By virtue of the identification (33), the Hopf algebra, $\text{Char-GL}$, of the universal character of $GL$ is immediately seen to be isomorphic to $\text{Symm}-\Lambda$. Its structure is well known (for references see [11, 13]) and some of its properties are summarized as follows.
Theorem 3.2: The ring of universal characters of GL is a graded self dual, bicommutative Hopf algebra, which we denote by Char-GL. Its structure maps are given by:

- **Product** 
  \[ m(\{\mu\} \otimes \{\nu\}) = \{\mu\} \cdot \{\nu\} = \{\mu \cdot \nu\} = \sum c_{\mu,\nu}^\lambda \{\lambda\} \]

- **Unit** 
  \[ \eta(1) = \{0\} \quad \text{with} \quad \{0\} \cdot \{\mu\} = \{\mu\} = \{\mu\} \cdot \{0\} \]

- **Coproduct** 
  \[ \Delta(\{\lambda\}) = \sum_{\mu,\nu} c_{\mu,\nu}^\lambda \{\mu\} \otimes \{\nu\} \]

- **Counit** 
  \[ \epsilon(\{\mu\}) = \langle \{0\} | \{\mu\} \rangle = \delta_{0,\mu} \]

- **Antipode** 
  \[ S(\{\lambda\}) = (-1)^{|\lambda|} \{\lambda'\} \]

- **Scalar product** 
  \[ \langle \cdot | \cdot \rangle(\{\mu\} \otimes \{\nu\}) = \langle \mu | \nu \rangle = \delta_{\mu,\nu}. \quad (38) \]

where the coefficients \( c_{\mu,\nu}^\lambda \) are the Littlewood-Richardson coefficients, \( \lambda' \) is the conjugate (transposed) partition and \( \langle \cdot | \cdot \rangle : \Lambda \otimes \Lambda \to \mathbb{Z} \) is the usual Schur-Hall scalar product in terms of which we have

**Self-duality** 
\[ \langle \Delta(\{\lambda\}) | \{\mu\} \otimes \{\nu\} \rangle = \langle \{\lambda\} | \{\mu\} \cdot \{\nu\} \rangle. \quad (39) \]

Because of its importance in what follows we map the antipode identity (19) of Symm-\( \Lambda \), into the antipode identity of Char-GL:

\[ \sum_{\nu} (-1)^{|\nu|} \{\lambda/\nu\} \cdot \{\nu'\} = \delta_{\lambda,0} \{0\}. \quad (40) \]

3.3 The orthogonal case

Having shown that the irreducible universal characters \([\lambda]\) of the orthogonal group O can be expressed in terms of universal characters of GL by \([\lambda] = \{\lambda/C\}\), it is possible to exploit infinite Schur function series and the Hopf algebra Char-GL to identify the action of the structure maps on the ring of characters \([\lambda]\) forming the canonical basis of Char-O. This action, as will be proved in the following section, takes the following form:

Theorem 3.3: The algebra Char-O generated by the universal characters \([\lambda]\) of the orthogonal
group $O$ is a bicommutative Hopf algebra. Its structure maps are given by:

- **Product**
  \[ m([\mu] \cdot [\nu]) = [\mu] \cdot [\nu] = \sum [\mu/\zeta \cdot \nu/\zeta] \]

- **Unit**
  \[ \eta(1) = [0] \text{ with } [0] \cdot [\mu] = [\mu] = [\mu] \cdot [0] \]

- **Coproduct**
  \[ \Delta([\lambda]) = \sum [\lambda/(\zeta B)] \otimes [\zeta] = \sum [\lambda/\zeta] \otimes [\zeta/B] \]

- **Counit**
  \[ \epsilon([\lambda]) = \sum_{\gamma \in C} (-1)^{|\gamma|/2} \delta_{\lambda,\gamma} = \delta_{\lambda,C} \]

- **Antipode**
  \[ S([\lambda]) = (-1)^{|\lambda|} [\lambda/(\zeta B)] \]

- **Scalar product**
  \[ \langle \cdot | \cdot \rangle_2([\mu] \otimes [\nu]) = \langle \mu | \nu \rangle_2 = \delta_{\mu,\nu}. \] (41)

### 3.4 The symplectic case

In the same way, by exploiting the fact that the irreducible (or indecomposable) universal characters \( \langle \lambda \rangle \) of the symplectic group $Sp$ can be expressed in terms of universal characters of $GL$ by \( \langle \lambda \rangle = \{\lambda/A\} \), we can identify the action of the structure maps on the ring of characters \( \langle \lambda \rangle \) forming the canonical basis of $Char\cdot Sp$. This action, as will be proved in the following section, takes the following form:

**Theorem 3.4**: The algebra $Char\cdot Sp$ generated by the universal characters \( \langle \lambda \rangle \) of the symplectic group $Sp$ is a bicommutative Hopf algebra. Its structure maps are given by:

- **Product**
  \[ m(\langle \mu \rangle \cdot \langle \nu \rangle) = \langle \mu \rangle \cdot \langle \nu \rangle = \sum \langle \mu/\zeta \cdot \nu/\zeta \rangle \]

- **Unit**
  \[ \eta(1) = \langle 0 \rangle \text{ with } \langle 0 \rangle \cdot \langle \mu \rangle = \langle \mu \rangle = \langle \mu \rangle \cdot \langle 0 \rangle \]

- **Coproduct**
  \[ \Delta(\langle \lambda \rangle) = \sum [\lambda/(\zeta B)] \otimes [\zeta] = \sum [\lambda/\zeta] \otimes [\zeta/B] \]

- **Counit**
  \[ \epsilon(\langle \lambda \rangle) = \sum_{\gamma \in C} (-1)^{|\gamma|/2} \delta_{\lambda,\gamma} = \delta_{\lambda,C} \]

- **Antipode**
  \[ S(\langle \lambda \rangle) = (-1)^{|\lambda|} [\lambda/(\zeta B)] \]

- **Scalar product**
  \[ \langle \cdot | \cdot \rangle_{11}(\langle \mu \rangle \otimes [\nu]) = \langle \mu | \nu \rangle_{11} = \delta_{\mu,\nu}. \] (42)

### 3.5 Directory of results

All the above results, and a considerable amount of additional information, regarding the three Hopf algebras of character rings, $Char\cdot GL$, $Char\cdot O$ and $Char\cdot Sp$, are gathered together in Table 1.
<table>
<thead>
<tr>
<th>Λ</th>
<th>Char-GL</th>
<th>Char-O</th>
<th>Char-Sp</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{0}</td>
<td>[0]</td>
<td>⟨0⟩</td>
</tr>
<tr>
<td>$p_n$</td>
<td>$\sum_{b=0}^{n-1} (-1)^b{n-b, 1^b}$</td>
<td>$\sum_{b=0}^{n-1} (-1)^b[n-b, 1^b] + \chi(2</td>
<td>n)[0]$</td>
</tr>
<tr>
<td>$h_n$</td>
<td>{n}</td>
<td>[n/D]</td>
<td>⟨n⟩</td>
</tr>
<tr>
<td>$e_n$</td>
<td>{1^n}</td>
<td>[1^n]</td>
<td>⟨1^n/B⟩</td>
</tr>
<tr>
<td>$s_\lambda$</td>
<td>{λ}</td>
<td>[λ/D]</td>
<td>⟨λ/B⟩</td>
</tr>
<tr>
<td>$o_\lambda$</td>
<td>{λ/C}</td>
<td>[λ]</td>
<td>⟨λ/(BC)⟩</td>
</tr>
<tr>
<td>$sp_\lambda$</td>
<td>{λ/A}</td>
<td>[λ/(AD)]</td>
<td>⟨λ⟩</td>
</tr>
</tbody>
</table>

| $m$ | $m(\{\mu\} \otimes \{\nu\}) = \{\mu \cdot \nu\}$ | $m([\mu] \otimes [\nu]) = \sum_{\zeta \in \mathbb{P}} [\mu/\zeta \cdot \nu/\zeta]$ | $m([\mu] \otimes [\nu]) = \sum_{\zeta \in \mathbb{P}} (\mu/\zeta \cdot \nu/\zeta)$ |
| $\Delta$ | $\Delta(\{\lambda\}) = \sum_{\zeta \in \mathbb{P}} \{\lambda/\zeta \} \otimes \{\zeta\}$ | $\Delta([\lambda]) = \sum_{\zeta \in \mathbb{P}} [\lambda/\zeta] \otimes [\zeta/D]$ | $\Delta([\lambda]) = \sum_{\zeta \in \mathbb{P}} (\lambda/\zeta) \otimes (\zeta/B)$ |
| $\epsilon$ | $\epsilon(\{\lambda\}) = \delta_{\lambda,0}$ | $\epsilon([\lambda]) = \sum_{\gamma \in \mathbb{C}} (-1)^{|\gamma|/2} \delta_{\lambda,\gamma}$ | $\epsilon([\lambda]) = \sum_{\alpha \in \mathbb{A}} (-1)^{|\alpha|/2} \delta_{\lambda,\alpha}$ |
| $\eta$ | $\eta(1) = \{0\}$ | $\eta(1) = [0]$ | $\eta(1) = (0)$ |
| $S$ | $S(\{\lambda\}) = (-1)^{|\lambda|} \{\lambda'\}$ | $S([\lambda]) = (-1)^{|\lambda|} [\lambda'/(AD)]$ | $S([\lambda]) = (-1)^{|\lambda|} \langle \lambda'/(BC) \rangle$ |

| $\langle \cdot | \cdot \rangle$ | $\langle \{\lambda\} | \{\mu\} \rangle = \delta_{\lambda,\mu}$ | $\langle [\lambda] | [\mu] \rangle_2 = \delta_{\lambda,\mu}$ | $\langle \langle \lambda \rangle | \langle \mu \rangle \rangle_{11} = \delta_{\lambda,\mu}$ |
The first column of this directory gives the abstract Hopf algebra notation for bases and morphisms of Symm-Λ, for any \( n \in \mathbb{N} \) and \( \lambda \in \mathcal{P} \). The second column gives the notion for the Hopf algebra of the universal character ring of the general linear group, as studied for example in [11]. The third and fourth columns provide the isomorphic images of the structure maps and bases in the character rings of the orthogonal and symplectic groups. We have used the notational convention whereby \( \chi(P) \) is the truth symbol, that is \( \chi(P) = 1 \) if the proposition \( P \) is true, and 0 otherwise. Thus \( \chi(2|n) = 1 \) if \( n \) is even and \( \chi(2|n) = 0 \) if \( n \) is odd.

**Remark.** While \( \Lambda \) and Char-GL share the same Schur-Hall scalar product we emphasise that Char-O and Char-Sp can quite naturally be equipped with new structure maps, plethystic Schur-Hall scalar products, indexed by 2 and 11, which are defined as shown in Table 1 so as to ensure that the orthogonal and symplectic Schur functions form orthonormal bases of Char-O and Char-Sp, respectively.

The precise definitions of the bases involved in some of the formulae of Table 1 will be given in the following sections. However, this table makes it clear that there are unique instances of symmetric functions, such as power sum symmetric functions, which are tied to the underlying alphabet and are, up to isomorphism, equivalent in all the character Hopf algebras under consideration. Despite this, if written in the canonical basis of a specific character Hopf algebra, it can be seen that such objects may look different and may also exhibit combinatorial differences.

### 4 Orthogonal and symplectic character ring Hopf algebras

In this section we provide proofs of the validity of each of the structure map formulae listed in Table 1.

#### 4.1 The case of Char-O

We consider in turn each of the structure maps listed in Theorem 3.3.

The product formula

\[
[\mu] \cdot [\nu] = \sum_{\zeta \in \mathcal{P}} [\mu/\zeta \cdot \nu/\zeta]
\]

(43)

is a classical result of Newell [28] and Littlewood [26] that appears as a special case of the development in [13] for more general subgroups of the general linear group. Its derivation can be accomplished most easily by noting that \( [\mu] \cdot [\nu] = \{\mu/C\} \cdot \{\nu/C\} = \left(\{\mu/C\} \cdot \{\nu/C\}\right)/D \)
where the coefficient of \( \{ \lambda \} \) in \( \langle \{ \lambda \} | (\{ \mu/C \} \cdot \{ \nu/C \})/D \rangle \) is given by

\[
\langle \{ \lambda \} | (\{ \mu/C \} \cdot \{ \nu/C \})/D \rangle = \langle \{ \lambda \} \cdot D | \{ \mu/C \} \cdot \{ \nu/C \} \rangle = \langle \Delta(\{ \lambda \} \cdot D) | \{ \mu/C \} \otimes \{ \nu/C \} \rangle \\
= \sum_{\zeta \in P} \langle \Delta(\{ \lambda \}) \cdot (D \otimes D) \cdot \{ \zeta \} \otimes \{ \zeta \} | \{ \mu/C \} \otimes \{ \nu/C \} \rangle \\
= \sum_{\zeta \in P} \langle \Delta(\{ \lambda \}) | \{ \mu/(C \zeta D) \} \otimes \{ \nu/(C \zeta D) \} \rangle \\
= \sum_{\zeta \in P} \langle \Delta(\{ \lambda \}) | \{ \mu/\zeta \} \otimes \{ \nu/\zeta \} \rangle = \sum_{\zeta \in P} \langle \{ \lambda \} | \{ \mu/\zeta \} \cdot \{ \nu/\zeta \} \rangle ,
\]

from which the result (43) follows.

To find the coproduct we need to find first the ordinary coproduct of a skew Schur function. This can be looked up in Macdonald [27] (Eq. 5.9 and 5.10, p72). The idea is to expand

\[
s^\lambda(x, y, z) = \sum_{\nu} s^\lambda/\nu(x, y) s^\nu(z)
\]

and comparing coefficients of \( s^\nu(z) \) gives

\[
s^\lambda/\nu(x, y) = \sum_{\mu} s^\lambda/\mu(x)s^\mu/\nu(y) , \quad \text{that is} \quad \Delta(s^\lambda/\nu) = \sum_{\mu} s^\lambda/\mu \otimes s^\mu/\nu .
\]

Now we can proceed to compute

\[
\Delta([\lambda]) : = \Delta(\{ \lambda/C \}) = \sum_{\gamma \in C} (-1)^{\gamma!}/2 \Delta(\{ \lambda/\gamma \}) = \sum_{\gamma \in C, \zeta \in P} (-1)^{\gamma!}/2 \{ \lambda/\zeta \} \otimes \{ \zeta/\gamma \} \\
= \sum_{\zeta \in P} \{ \lambda/\zeta \} \otimes [\zeta] = \sum_{\zeta \in P} ([\lambda/(\zeta D)] \otimes [\zeta] = \sum_{\zeta \in P} [\lambda/(\zeta D)] \otimes [\zeta] .
\]

This can equally well be rewritten to give a second form of the coproduct derived using a pair of related expansions of a skew Schur function, \( s^\lambda/\zeta = \sum_\sigma c^\lambda_{\zeta, \sigma} s^\sigma \) and \( s^\lambda/\sigma = \sum_\zeta c^\lambda_{\sigma, \zeta} s^\zeta \), a move we use below frequently. Here it gives,

\[
\Delta([\lambda]) = \sum_{\zeta \in P} ([\lambda/(\zeta D)] \otimes [\zeta] = \sum_{\zeta \in P} ([\lambda/\zeta D] \otimes [\zeta] \\
= \sum_{\zeta, \sigma \in P} c^\lambda_{\zeta, \sigma} [\sigma/D] \otimes [\zeta] = \sum_{\sigma \in P} [\sigma/D] \otimes [\lambda/\sigma] .
\]

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The coproduct $\Delta([\lambda])$ is cocommutative, as can be seen by using in the same way as above the connection between the outer product of Schur functions $s_\zeta \cdot s_\delta = \sum_\eta c_{\zeta,\delta}^\eta s_\eta$ and the skew Schur functions expansion $s_{\sigma/\delta} = \sum_\zeta c_{\zeta,\delta}^\sigma s_\zeta$, to obtain a third form:

$$
\Delta([\lambda]) = \sum_{\zeta \in P} \left[ \frac{\lambda}{(\zeta D)} \right] \otimes [\zeta] = \sum_{\zeta,\sigma \in P, \delta \in D} \left[ \frac{\lambda}{\sigma} \right] \otimes \left[ \frac{\sigma}{\delta} \right] = \sum_{\sigma \in P} [\lambda/\sigma] \otimes [\sigma/D].
$$

Finally, we may use $s_{\lambda/\sigma} = \sum_\zeta c_{\lambda,\sigma}^\zeta s_\zeta$ and $\sum_\sigma c_{\lambda,\sigma}^\zeta s_\sigma = s_{\lambda/\zeta}$ to obtain the fourth form

$$
\Delta([\lambda]) = \sum_{\sigma \in P} [\lambda/\sigma] \otimes [\sigma/D] = \sum_{\sigma,\zeta \in P} c_{\lambda,\zeta}^\sigma [\zeta] \otimes [\sigma/D]
\quad = \sum_{\zeta \in P} [\zeta] \otimes \left[ \frac{\lambda}{(\zeta D)} \right] = \sum_{\zeta \in P} \left[ \frac{\lambda}{\zeta} \right] \otimes [\lambda/D].
$$

The actions of the countit, $\epsilon$, the unit, $\eta$, and the antipode, $S$, follow immediately from their action in $\text{Char-GL}$ and the fact that $[\lambda] = \{\lambda/C\}$. Thus in the $\text{Char-O}$ basis

$$
\epsilon([\lambda]) = \epsilon(\{\lambda/C\}) = \sum_{\gamma \in \mathcal{C}} (-1)^{\gamma|/2} \epsilon(\{\lambda/\gamma\}) = \sum_{\gamma \in \mathcal{C}} (-1)^{\gamma|/2} \delta_{\lambda,\gamma};
\quad \eta(1) = \{0\} = \{0/D\} = \{0\};
\quad S(\{\lambda\}) = S(\{\lambda/C\}) = \sum_{\gamma \in \mathcal{C}} (-1)^{\gamma|/2} S(\{\lambda/\gamma\}) = \sum_{\gamma \in \mathcal{C}} (-1)^{\gamma|/2} (-1)^{|\lambda|-|\gamma|} \{\lambda/\gamma'\}
\quad = (-1)^{|\lambda|} \{\lambda/C'\} = (-1)^{|\lambda|} \{\lambda/A\} = (-1)^{|\lambda|} \{\lambda/(A/D)\} = (-1)^{|\lambda|} \{\lambda/(AD)\}
$$

all as shown in Table 1.

By exploiting the Schur-Hall scalar product we may reinterpret $\epsilon$ and introduce its convolutive inverse $\epsilon^{-1}$ as follows:

**Definition 4.5:** The countit $\epsilon$ and its convolutive inverse $\epsilon^{-1}$ for $\text{Char-O}$ may be interpreted as linear forms $c$ and $d: \text{Char-GL} \to \mathbb{Z}$ defined as follows:

$$
\epsilon([\lambda]) = c(\{\lambda\}) \quad \text{with} \quad c(\{\lambda\}) := \{C \mid \{\lambda\}\} = \sum_{\gamma \in \mathcal{C}} (-1)^{\gamma|/2} \{\gamma\} \mid \{\lambda\} = \sum_{\gamma \in \mathcal{C}} (-1)^{\gamma|/2} \delta_{\gamma,\lambda};
\quad \epsilon^{-1}(\{\lambda\}) = d(\{\lambda\}) \quad \text{with} \quad d(\{\lambda\}) := \{D \mid \{\lambda\}\} = \sum_{\delta \in \mathcal{D}} \{\delta\} \mid \{\lambda\} = \sum_{\delta \in \mathcal{D}} \delta_{\delta,\lambda}.
$$

**Corollary 4.6:** (see [11]) The linear forms (1-cochains) $c$ and $d$ are convolutive inverses with respect to the $\text{Char-GL}$ outer coproduct and product in $\mathbb{Z}$. 

\footnote{This definition should be compared with a slightly different point of view developed in the section on adapted normal ordered products in [5], which can be used to define a quantum field theory on an external background.}
Proof:
\[
(c \star d)(\{\lambda\}) = \sum_{(\lambda)} c(\{\lambda_{(1)}\})d(\{\lambda_{(2)}\}) = \sum_{(\lambda)} \langle C \mid \{\lambda_{(1)}\}\rangle\langle D \mid \{\lambda_{(2)}\}\rangle
\]
\[
\sum_{(\lambda)} (C \otimes D \mid \{\lambda_{(1)}\} \otimes \{\lambda_{(2)}\}) = \langle C \otimes D \mid \Delta(\{\lambda\})\rangle
\]
\[
= \langle CD \mid \{\lambda\}\rangle = (1 \mid \{\lambda\}) = \epsilon(\{\lambda\}) = \delta_{\lambda,0}.
\] (53)

Finally, we might check that the product and coproduct are mutual coalgebra and algebra homomorphisms. We establish this fact by direct computation:
\[
(\Delta m)(\lambda \otimes \mu) = \sum_{\zeta} \Delta([\lambda/\zeta \cdot \mu/\zeta]) = \sum_{\rho,\zeta} [\lambda/(\zeta \cdot \rho/\zeta)] \otimes [\rho/D]
\]
\[
= \sum_{\sigma,\rho,\zeta} [\lambda/(\zeta/\sigma) \cdot \mu/(\zeta/\sigma)] \otimes [\rho/D] = \sum_{\zeta,\sigma,\zeta} [\lambda/(\zeta/\sigma) \cdot \mu/(\zeta/\sigma)] \otimes [(\zeta/\sigma)/D]
\]
\[
= \sum_{\tau,\xi,\tau,\xi} [\lambda/(\zeta/\tau) \cdot \xi/(\tau/\xi)] \otimes [\sigma/(\tau D) \cdot \xi/(\tau D)] = \sum_{\sigma,\xi} [\lambda/(\sigma/\tau) \cdot \mu/(\xi/\tau)] \otimes [(\sigma/\tau D) \cdot [\xi/D]]
\]
\[
= \sum_{\sigma,\xi} (\lambda/\sigma) \cdot [(\mu/\xi) \otimes [\xi/D]] = \Delta([\lambda]) \cdot \Delta([\mu]) = m(\Delta([\lambda]) \otimes \Delta([\mu]))
\] (54)
showing the claim.

Remarks. It could be argued that the above proof is unnecessary. We considered just a linear isomorphism on the module underlying the symmetric function Hopf algebra, and the result is in a natural way, a homomorphic image. However, the displayed calculations show explicitly how the structure maps are written in the orthogonal Schur function bases, how the combinatorics alters, and that everything is set up correctly.

Note also the most remarkable fact that the structure of the Hopf algebra Char-O does not distinguish between even and odd orthogonal groups. It does not even rely on the fact that the metric tensor $g_{ij} = g_{ji}$ of Schur symmetry type $\{2\}$, which defines the orthogonal group, is invertible. Such degenerate cases are instances of Cayley-Klein groups (see conclusions for further comments). The even or oddness of the underlying group will show up in a subtle way when we define particular bases for these Hopf algebras below.

4.2 The case of Char-\(\text{Sp}\)

The validity of the structure maps of Char-\(\text{Sp}\) given in Theorem 3.4 may be established by copying and pasting the proof for the orthogonal case. One merely changes all orthogonal characters into symplectic ones, $[\lambda] \rightarrow \langle \lambda \rangle$, and interchanges Schur function series, $C \leftrightarrow A$ and $D \leftrightarrow B$. All arguments run through as before. In the case of the counit, it is also necessary to interchange the labelling on the linear forms, $c \rightarrow a$ and $d \rightarrow b$, where by analogy with Definition 4.5 we have:
Definition 4.7: The counit $\epsilon$ and its convolutive inverse $\epsilon^{-1}$ for Char-Sp may be interpreted as linear forms $a$ and $b$: $\text{Char-GL} \to \mathbb{Z}$ defined as follows:

$$\epsilon(\langle \lambda \rangle) = a(\{\lambda\}) \text{ with } a(\{\lambda\}) := \langle A \mid \{\lambda\} \rangle = \sum_{\alpha \in A} (-1)^{|\alpha|/2} \langle \{\alpha\} \mid \{\lambda\} \rangle = \sum_{\alpha \in A} (-1)^{|\alpha|/2} \delta_{\alpha,\lambda}, \quad (55)$$

$$\epsilon^{-1}(\langle \lambda \rangle) = b(\{\lambda\}) \text{ with } b(\{\lambda\}) := \langle B \mid \{\lambda\} \rangle = \sum_{\beta \in B} \langle \{\beta\} \mid \{\lambda\} \rangle = \sum_{\beta \in B} \delta_{\beta,\lambda}. \quad (56)$$

Once again as in Corollary 4.6 we have:

Corollary 4.8: (see [11]) The linear forms (1-cochains) $a$ and $b$ are convolutive inverses with respect to the Char-GL outer coproduct and product in $\mathbb{Z}$.

5 Bases for Char-O and Char-Sp

5.1 Power sum symmetric functions

A major issue in setting the above abstract machinery to work in concrete (physical) examples, is a proper identification in the various character rings of the usual canonical bases of the symmetric function ring. In making this identification, we will encounter some familiar and also some surprising results. We start with the power sum symmetric functions on a finite number of variables $N$. The one part power sum symmetric functions are defined on the variables $(x_1, \ldots, x_N)$ by

$$p_n := \sum_{i=1}^{N} x_i^n \quad (56)$$

which is independent of the meaning of the alphabet.

In the $\text{GL}(N)$ case the $x_i$ are the eigenvalues of a $\text{GL}(N)$ element $g$ within a $\text{GL}(N)$ conjugacy class. There is the constraint $\prod_i x_i \neq 0$ in force to ensure invertibility. We use the well known hook expansion in terms of the Schur functions identified with irreducible $\text{GL}(N)$ characters:

$$p_n(x_1, \ldots, x_n) = \sum_{a+b+1=n} (-1)^b \{a + 1, 1^b\}(x_1, \ldots, x_N) \quad (57)$$

This formula is stable with respect to the limit $N \to \infty$ so that we immediately have in the case of $\text{Char-GL}$ the identification

$$p_n = \sum_{a+b+1=n} (-1)^b \{a + 1, 1^b\}. \quad (58)$$

In branching to orthogonal or the symplectic groups, as one can see from (119) the eigenvalues now generally speaking come in pairs $x_k$ and $\overline{x}_k$ and we can split $p_n$ into at least two
parts. In the orthogonal $O(N)$ case, there are four possibilities, and in the symplectic $Sp(N)$ case there are two. Confining attention to the unimodular case it follows from (119) that:

\[
p_n(x, \overline{\mathcal{X}}) = p_n(x) + p_n(\overline{x}) \quad \text{for } SO(2K);
\]
\[
p_n(x, \overline{\mathcal{X}}, 1) = p_n(x) + p_n(\overline{x}) + 1 \quad \text{for } SO(2K + 1);
\]
\[
p_n(x, \overline{\mathcal{X}}) = p_n(x) + p_n(\overline{x}) \quad \text{for } Sp(2K);
\]
\[
p_n(x, \overline{\mathcal{X}}, 1) = p_n(x) + p_n(\overline{x}) + 1 \quad \text{for } Sp(2K + 1),
\]

where in each case $p_n(x) = \sum_{i=1}^{K} x_i^n$ and $p_n(\overline{x}) = \sum_{i=1}^{K} \overline{x}_i^n$.

Clearly, this is the place where the dimensionality $N = 2K$ or $N = 2K + 1$ comes into play. However, this does not prevent us from establishing a result stable in the $K \to \infty$ limit. Indeed we find as a corollary to (58) the result appropriate to Char-O:

**Corollary 5.9:**

\[
p_0 = [0] \quad \text{and} \quad p_n = \sum_{a+b+1=\nu} (-1)^b [a+1, 1^b] + \chi(2|n)[0] \quad \text{for } n \geq 1,
\]

where $\chi$ is the truth function, so that $\chi(2|n) = 1$ if $n$ is even and $\chi(2|n) = 0$ if $n$ is odd. 

Note that the $n$ in this truth function has to do with the index of the one part power sums, and not with the number of its variables!

**Proof:** For $n = 0, 1$ we can directly verify that $p_0 = [0]$ and $p_1 = [1]$, thereby proving the statement in these cases. Henceforth we assume $n \geq 2$. The $\mathcal{D}$ series partitions $\delta \in \mathcal{D} = 2\mathcal{P}$ have only even parts, and of these only the partitions of type $\{2k\}$ can fit into a hook. Thus

\[
p_n = \sum_{a+b+1=\nu} (-1)^b \{a+1, 1^b\} = \sum_{b \geq 0} (-1)^b \{n-b, 1^b\}
\]
\[
= \sum_{b \geq 0} (-1)^b [(n-b, 1^b)/D] = \sum_{k \geq 0} \sum_{b \geq 0} (-1)^b [(n-b, 1^b)/(2k)]
\]
\[
= \sum_{b \geq 0} (-1)^b [(n-b, 1^b)] + \sum_{k \geq 1} \sum_{b \geq 0} (-1)^b [(n-b-2k) \cdot (1^b)]
\]
\[
= \sum_{b \geq 0} (-1)^b [(n-b, 1^b)] + \sum_{k \geq 1} \sum_{\zeta \in \mathcal{P}} (-1)^{\lvert k \rvert} [(n-2k)/\zeta \cdot \zeta']
\]
\[
= \sum_{b \geq 0} (-1)^b [(n-b, 1^b)] + \sum_{k \geq 1} \delta_{n-2k,0}[0]
\]
\[
= \sum_{b \geq 0} (-1)^b [(n-b, 1^b)] + \chi(2|n)[0],
\]

(61)
where, in the penultimate line, the second term has resulted from the antipode property (40). □

We have an exactly analogous result for \text{Char-Sp}:

**Corollary 5.10:**

\[
p_0 = \langle 0 \rangle \quad \text{and} \quad p_n = \sum_{a+b+1=n} (-1)^b \langle a+1, b \rangle + \chi(2|n)\langle 0 \rangle \quad \text{for } n \geq 1.
\] (62)

We recall that the one part power sums are the primitive elements of the symmetric function Hopf algebra. They form a rational basis of this Hopf algebra. This implies:

**Proposition 5.11:** The one part power sums \( p_n \) map to the primitive elements of the Hopf algebra of the universal character rings of GL, O and Sp. That is, in each case we have

\[
\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n.
\] (63)

**Proof:** This is a trivial consequence of (20), since the isomorphism of Hopf algebras which we have established is independent of the underlying alphabet, and hence does not alter the coproduct properties of the power sums. □

### 5.2 Complete symmetric functions

In this section we investigate the nature of complete symmetric functions in each of our three rings of universal characters by means of the maps between Schur functions and the characters.

First, our maps allow us to see immediately that, in accordance with the formulae of Table 1, we have

\[
h_n = s_n = \{ n \},
\]

\[
h_n = s_n = \{ n \} = [n/D] = \sum_k [n/(2k)] = \sum_k [n - 2k],
\]

\[
h_n = s_n = \{ n \} = \langle n/B \rangle = \langle n \rangle,
\] (64)

where \([n/2]\) is the integer part of \(n/2\). Moreover, we have

**Proposition 5.12:** The above images of the one part complete symmetric functions \( h_n \) under the maps from the Hopf algebra of symmetric functions to the universal character rings of GL, O and Sp are divided powers [29, 3, 36], their coproducts take the form:

\[
\Delta(h_n) = \sum_r h_{n-r} \otimes h_r.
\] (65)
Proof: These results are a direct consequence of (20), since the maps between the Hopf algebras are isomorphisms, but they can also be derived as follows.

\[ \Delta(\{n\}) = \sum_{\zeta} \{n/\zeta\} \otimes \{\zeta\} = \sum_{r} \{n/r\} \otimes \{r\} = \sum_{r} \{n - r\} \otimes \{r\}, \]

\[ \Delta([n/D]) = \sum_{\zeta} [n/(\zeta D)] \otimes [\zeta/D] = \sum_{r} [(n/r)/D] \otimes [r/D] = \sum_{r} [(n - r)/D] \otimes [r/D], \]

\[ \Delta(\langle n \rangle) = \sum_{\zeta} \langle n/\zeta \rangle \otimes \langle \zeta/B \rangle = \sum_{r} \langle n/r \rangle \otimes \langle r/B \rangle = \sum_{r} \langle n - r \rangle \otimes \langle r \rangle. \quad (66) \]

5.3 Elementary symmetric functions

The elementary symmetric functions \( e_n \) map as follows to the three character rings of interest:

\[ e_n = s_{1n} = \{1^n\}, \]
\[ e_n = s_{1n} = \{1^n\} = [1^n/D] = [1^n], \]
\[ e_n = s_{1n} = \{1^n\} = \langle 1^n/B \rangle = \sum_{r} \langle 1^{n-2r} \rangle = \sum_{r=0}^{[n/2]} \langle 1^{n-2r} \rangle. \quad (67) \]

Moreover, we have

**Proposition 5.13**: The above images of the one part elementary symmetric functions \( e_n \) under the maps from the Hopf algebra of symmetric functions to the universal character rings of \( \text{GL} \), \( \text{O} \) and \( \text{Sp} \) are again divided powers since their coproducts all take the form:

\[ \Delta(e_n) = \sum_{r} e_{n-r} \otimes e_r. \quad (68) \]

Proof: These results are a direct consequence of (20), since the maps between the Hopf algebras are isomorphisms, but they can also be derived as follows.

\[ \Delta(\{1^n\}) = \sum_{\zeta} \{1^n/\zeta\} \otimes \{\zeta\} = \sum_{r} \{1^n/1^r\} \otimes \{1^r\} = \sum_{r} \{1^{n-r}\} \otimes \{1^r\}, \]

\[ \Delta([1^n]) = \sum_{\zeta} [1^n/\zeta] \otimes [\zeta/D] = \sum_{r} [1^n/1^r] \otimes [1^r/D] = \sum_{r} [1^{n-r}] \otimes [1^r], \quad (69) \]

\[ \Delta(\langle 1^n/B \rangle) = \sum_{\zeta} \langle 1^n/(\zeta B) \rangle \otimes \langle \zeta/B \rangle = \sum_{r} \langle (1^n/1^r)/B \rangle \otimes \langle 1^r/B \rangle \]
\[ = \sum_{r} \langle 1^{n-r}/B \rangle \otimes \langle 1^r/B \rangle. \]
6 Scalar products, adjoints, Foulkes derivatives and duals

6.1 Scalar products

Unlike the Schur functions of general linear type, it can be readily checked, that Schur functions of orthogonal type \( o_\lambda = [\lambda] \) and of the symplectic type \( sp_\lambda = \langle \lambda \rangle \) are not orthogonal with respect to the Schur-Hall scalar product. It is hence necessary to define new 'orthogonal' and 'symplectic' scalar products, accounting for the fact that we consider the orthogonal and symplectic Schur functions to be universal characters of irreducible orthogonal and symplectic group representations. The orthogonality of the universal characters of \( \text{Char-GL}, \text{Char-O} \) and \( \text{Char-Sp} \) are expressed through the following:

**Definition 6.14:** The general linear, orthogonal and symplectic scalar products are defined by:

\[
\langle \cdot | \cdot \rangle : \text{Char-GL} \otimes \text{Char-GL} \to \mathbb{Z} \quad \text{with} \quad \langle \{ \lambda \} | \{ \mu \} \rangle = \delta_{\lambda,\mu};
\]
\[
\langle \cdot | \cdot \rangle_2 : \text{Char-O} \otimes \text{Char-O} \to \mathbb{Z} \quad \text{with} \quad \langle [\lambda] | [\mu] \rangle_2 = \delta_{\lambda,\mu};
\]
\[
\langle \cdot | \cdot \rangle_{11} : \text{Char-O} \otimes \text{Char-O} \to \mathbb{Z} \quad \text{with} \quad \langle \langle \lambda \rangle | \langle \mu \rangle \rangle_{11} = \delta_{\lambda,\mu},
\]
for all partitions \( \lambda \) and \( \mu \) \hspace{1cm} \Box

The indices 2 and 11 are a reminder of the plethystic character of the branching from GL to O (see [13] and the previous introductory remarks).

The relation between the scalar products of \( \text{Char-O} \) and \( \text{Char-Sp} \) and those of \( \text{Char-GL} \) is such that

\[
\langle [\lambda] | [\mu] \rangle_2 = \delta_{\lambda,\mu} = \langle \{ \lambda \} | \{ \mu \} \rangle = \langle [\lambda/D] | [\mu/D] \rangle;
\]
\[
\langle \langle \lambda \rangle | \langle \mu \rangle \rangle_{11} = \delta_{\lambda,\mu} = \langle \{ \lambda \} | \{ \mu \} \rangle = \langle \langle \lambda/B \rangle | \langle \mu/B \rangle \rangle.
\]

(71)

We now consider two maps from the ring of symmetric functions \( \Lambda \) into the ring \( \text{End}(\Lambda) \) and their general linear, orthogonal and symplectic counter parts. These are the operators, \( \cdot \) and \( \perp \) : \( \Lambda \to \text{End}(\Lambda) \) corresponding to 'multiplying by a Schur function' and its adjoint 'skewing with a Schur function', which we have used frequently above.

\[
s_\lambda \cdot (s_\mu) = \sum_\nu c^\lambda_{\mu,\nu} s_\nu = s_{\lambda\cdot\mu} \quad \text{and} \quad s_\lambda^\perp (s_\mu) = \sum_\nu c^\lambda_{\mu,\nu} s_\nu = s_{\mu/\lambda}.
\]

(72)

These two operations are related via the Schur-Hall scalar product

\[
\langle s_\mu \cdot (s_\nu) | s_\lambda \rangle = \langle s_\mu \cdot (s_\nu) | s_\lambda \rangle = c^\lambda_{\mu,\nu} = \langle s_\nu | s_{\lambda/\mu} \rangle = \langle s_\nu | s^\perp_\mu (s_\lambda) \rangle.
\]

(73)

Formule analogous to this exist for all our classical group universal characters:

**Corollary 6.15:** For all \( \lambda, \mu, \nu \in \mathcal{P} \)

\[
\langle \{ \mu \cdot \nu \} | \{ \lambda \} \rangle = c^\lambda_{\mu,\nu} = \langle \{ \nu \} | \{ \lambda/\mu \} \rangle;
\]
\[
\langle [\mu \cdot \nu] | [\lambda] \rangle_2 = c^\lambda_{\mu,\nu} = \langle [\nu] | [\lambda/\mu] \rangle_2;
\]
\[
\langle \langle \mu \cdot \nu \rangle | \langle \lambda \rangle \rangle_{11} = c^\lambda_{\mu,\nu} = \langle \langle \nu \rangle | \langle \lambda/\mu \rangle \rangle_{11}.
\]

(74)
Proof: The first of these follows from (73) through the usual identification \( \{ \lambda \} = s_\lambda \) for all \( \lambda \). For the second, one merely notes that from (70)

\[
\langle [\lambda] | [\mu \cdot \nu] \rangle_2 = \sum_\zeta c^\zeta_{\mu,\nu} \langle [\lambda] | [\zeta] \rangle_2 = c^\lambda_{\mu,\nu};
\]

\[
\langle [\lambda/\mu] | [\nu] \rangle_2 = \sum_\zeta c^\zeta_{\mu,\nu} \langle [\zeta] | [\nu] \rangle_2 = c^\lambda_{\mu,\nu} . \tag{75}
\]

The third is derived in an analogous manner.

However it should be noted that these relations do not help us identify an adjoint of multiplication for \( \text{Char-O} \) and \( \text{Char-Sp} \) since \([\mu] \cdot [\nu] \neq [\mu \cdot \nu] \) and \( \langle \mu \rangle \cdot \langle \nu \rangle \neq \langle \mu \cdot \nu \rangle \).

The adjoint of multiplication by a Schur function with respect to the Schur-Hall scalar product, that is the skew or \( \perp \), is called the Foulkes derivative. This can be used to introduce differential operators, for example in Macdonald [27] one finds both

\[
p_n^\perp = n \frac{\partial}{\partial p_n} \quad \text{and} \quad p_n^\perp = \sum_{r \geq 0} h_r \frac{\partial}{\partial h_{n+r}}. \tag{76}
\]

This leads to the interesting fact, that the coproduct can be written in terms of the adjoint:

\[
\Delta(f) = \sum_\mu s_{\mu}(f) \otimes s_\mu = \sum_{\mu(f)} \epsilon(s_{\mu}^\perp f(1)) f(2) \otimes s_\mu, \tag{77}
\]

and fulfils a Leibnitz type formula:

\[
s_{\lambda}^\perp(f g) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_{\mu}^\perp(f) s_{\nu}^\perp(g), \tag{78}
\]

justifying the name derivative. It is furthermore a rather important fact, that using the identification \( \pi_0 = 1, \pi_n = p_n \) and \( \pi_{-n} = n \partial/\partial p_n \) one easily checks that these operators generate the Heisenberg Lie algebra

\[
[\pi_n, \pi_m] = n \delta_{n+m,0} \pi_0 , \tag{79}
\]

closely related to vertex operators and the Witt, and Virasoro algebras used in string theory.

The main point we make in this section is to exemplify that in the case of the character ring Hopf algebras of the classical groups, the notion of the adjoint of multiplication and that of the Foulkes derivative need no longer be identical; they are logically distinct. Therefore we need new notation, and we choose to write \( \dagger \) for the adjoint, and keep \( \perp \) for the Foulkes derivative.

6.2 Adjoint of multiplication

Theorem 6.16: The adjoints of multiplication in \( \text{Char-GL} \), \( \text{Char-O} \) and \( \text{Char-Sp} \) with respect to the general linear, orthogonal and symplectic Schur-Hall scalar products are defined to be such
that:

\[
\langle \{ \nu \} \mid \{ \mu \}^\dagger(\{ \lambda \}) \rangle = \langle \{ \mu \} \cdot \{ \nu \} \mid \{ \lambda \} \rangle;
\]

\[
\langle [\nu] \mid [\mu]^\dagger([\lambda]) \rangle_2 = \langle [\mu] \cdot [\nu] \mid [\lambda] \rangle_2;
\]

\[
\langle (\nu) \mid (\mu)^\dagger((\lambda)) \rangle_{11} = \langle (\mu) \cdot (\nu) \mid (\lambda) \rangle_{11},
\]

(80)

respectively, for all partitions \( \lambda, \mu \) and \( \nu \). The action of these adjoints then take the explicit forms:

\[
\{ \mu \}^\dagger(\{ \lambda \}) = \{ \lambda/\mu \}; \quad [\mu]^\dagger([\lambda]) = [\mu] \cdot [\lambda]; \quad (\mu)^\dagger((\lambda)) = (\mu) \cdot (\lambda).
\]

(81)

Proof: We compute both sides of the requirements (80) separately using (81) on the left hand side. First in the general linear case we have

\[
\langle \{ \nu \} \mid \{ \mu \}^\dagger(\{ \lambda \}) \rangle = \sum_{\sigma} c^\lambda_{\mu,\sigma} \langle \{ \nu \} \mid \{ \sigma \} \rangle = c^\lambda_{\mu,\nu},
\]

\[
\langle \{ \mu \} \cdot \{ \nu \} \mid \{ \lambda \} \rangle = \sum_{\rho} c^\rho_{\mu,\nu} \langle \{ \rho \} \mid \{ \lambda \} \rangle = c^\lambda_{\mu,\nu},
\]

(82)

so that the two sides are identical as required. In the orthogonal case we have

\[
\langle [\nu] \mid [\mu]^\dagger([\lambda]) \rangle_2 = \sum_{\rho} \langle [\nu] \mid [\mu] \cdot [\lambda] \rangle_2 = \sum_{\rho} \langle [\nu] \mid [\mu/\rho \cdot \lambda/\rho] \rangle_2
\]

\[
= \sum_{\rho,\sigma,\tau,\eta} c^\rho_{\mu,\sigma} c^\lambda_{\rho,\tau} c^\eta_{\sigma,\tau} \langle [\nu] \mid [\eta] \rangle_2 = \sum_{\rho,\sigma,\tau} c^\mu_{\rho,\sigma} c^\lambda_{\rho,\tau} c^\nu_{\sigma,\tau},
\]

\[
\langle [\mu] \cdot [\nu] \mid [\lambda] \rangle_2 = \sum_{\sigma} \langle [\mu] \cdot [\nu] \mid [\sigma] \rangle_2
\]

\[
= \sum_{\sigma,\rho,\tau,\eta} c^\mu_{\sigma,\rho} c^\nu_{\sigma,\tau} c^\lambda_{\rho,\tau} \langle [\eta] \mid [\lambda] \rangle_2 = \sum_{\rho,\sigma,\tau} c^\mu_{\sigma,\rho} c^\nu_{\sigma,\tau} c^\lambda_{\rho,\tau}.
\]

(83)

The symmetry \( c^\mu_{\rho,\sigma} = c^\mu_{\sigma,\rho} \), then immediately yields equality, as required. An entirely analogous proof applies in the symplectic case.

Remark. We are thus left with the fact, that multiplication is a selfadjoint operation in \textup{Char-O} and in \textup{Char-Sp} with respect to the orthogonal and symplectic scalar products, respectively. In terms of group representations this amounts to saying that one can use the second rank tensor \( g_{ij} = g_{ji} \) of symmetry type \{2\} or \( f_{ij} = -f_{ji} \) of symmetry type \{11\} to raise or lower indices. Co- and contra-variant representations of the same index symmetry type are hence isomorphic. \( \square \)
6.3 Foulkes derivative

To find the correct Foulkes derivative, we exploit both comultiplication and the Schur-Hall scalar product in defining any \( a^\perp \) as follows

**Definition 6.17:** The Foulkes derivative is defined in an invariant way as

\[
a^\perp(b) = \langle a | b_{(1)} b_{(2)} \rangle .
\]  

(84)

It is easy to check that this definition is equivalent to the skew in the ordinary Symm-\( \Lambda \) case.

\[
s^\perp_\lambda(s_\mu) = \sum_\zeta \langle s_\lambda | s_\zeta \rangle s_{\mu/\zeta} = \sum_\zeta \delta_{\lambda\zeta} s_{\mu/\zeta} = s_{\mu/\lambda} .
\]

(85)

Furthermore, this definition can be written down in any character Hopf algebra where we have defined a Schur-Hall scalar product which represents the orthogonality of irreducible (indecomposable) characters.

**Corollary 6.18:** The Foulkes derivatives in the case of Char-GL, Char-O and Char-Sp are given by:

\[
\begin{align*}
(s_\lambda)^\perp(s_\mu) &= \{\lambda\}^\perp(\{\mu\}) = \{\mu/\lambda\} ; \\
(o_\lambda)^\perp(o_\mu) &= [\lambda]^\perp([\mu]) = [\mu/(\lambda D)] ; \\
(sp_\lambda)^\perp(sp_\mu) &= \langle \lambda \rangle^\perp(\langle \mu \rangle) = \langle \mu/(\lambda B) \rangle .
\end{align*}
\]  

(86)

**Proof:** The Hopf algebra definition for the Foulkes derivative is basis free, but depends on the scalar product, so that rephrasing (84) in the case of general linear, orthogonal and symplectic characters yields

\[
\begin{align*}
\{\lambda\}^\perp(\{\mu\}) &= \langle \{\lambda\} | \{\mu_{(1)}\} \{\mu_{(2)}\} \rangle = \sum_\zeta \langle \{\lambda\} | \{\zeta\} \{\mu/\zeta\} \rangle , \\
[\lambda]^\perp([\mu]) &= \langle [\lambda] | [\mu_{(1)}] [\mu_{(2)}] \rangle = \sum_\zeta \langle [\lambda] | [\zeta] \{\mu/\zeta D\} \rangle , \\
\langle \lambda \rangle^\perp(\langle \mu \rangle) &= \langle \langle \lambda \rangle | \{\mu_{(1)}\} \rangle_{11} \langle \mu_{(2)} \rangle = \sum_\zeta \langle \langle \lambda \rangle | \langle \zeta \rangle \rangle_{11} \langle \mu/(\zeta B) \rangle .
\end{align*}
\]  

(87)

where in the case of the orthogonal characters the fourth form of the coproduct given in (50) has been used, and its analogue in the case of the symplectic characters.

It is well known that the above definition (84) defines a derivation if the element \( a \) is a primitive element in the dual Hopf algebra \([10, 9]\). Here applying (84) in the case of a primitive element \( (m \geq 1) \) of Char-O we have

\[
p^\perp_n(p_m) = \langle p_n | p_m \rangle_2 + \langle p_n | [0] \rangle_2 p_m
\]

(88)
a consequence of (63). However using (61) we have

\[ \langle p_n | p_m \rangle_2 = \langle \sum_{b=0}^{n-1} (-1)^b[n - b, 1^b] + \chi(2|n)[0] | \sum_{d=0}^{m-1} (-1)^d[m - d, 1^d] + \chi(2|m)[0] \rangle_2 \]

\[ = n \delta_{n,m} + \chi(2|n)\chi(2|m), \]  

(89)

and

\[ \langle p_n | [0] \rangle_2 = \langle \sum_{b=0}^{n-1} (-1)^b[n - b, 1^b] + \chi(2|n)[0] | [0] \rangle_2 = \chi(2|n). \]  

(90)

Combining these results gives

\[ p_n^\dagger(p_m) = n\delta_{n,m} + \chi(2|m)\chi(2|n) + \chi(2|n)p_m. \]  

(91)

**Remark.** This, and an identical result in the symplectic case, shows that the power sum basis is not orthogonal with respect to the orthogonal or symplectic Schur-Hall scalar products. Furthermore, due to the different Hopf algebra structures of $H$ and $H^*$, the power sums $p_n$ are not the primitive elements of $H^*$. Hence the identification $p_n^\dagger = n\partial/\partial p_n$ of (76) that applies in the GL case fails to hold in the O and Sp cases. The correct way to introduce such (formal) derivatives would be to detect the primitive elements of $H^*$ and to find their dual basis under the relevant Schur-Hall scalar product. After this identification one could set up orthogonal and symplectic Heisenberg Lie algebras quite distinct from (79). This is, however, beyond the scope of the present paper. 

6.4 The dual Hopf algebras

In fact neither Char-O nor Char-Sp are self-dual Hopf algebras with respect to either the Char-GL Schur-Hall scalar product or the Char-O, respectively Char-Sp, scalar product. This shows that notwithstanding the Hopf algebra isomorphisms between Char-GL and both Char-O and Char-Sp, these latter Hopf algebras are not identical to Char-GL since, unlike Char-GL they are not self-dual. Since we will typically consider products such as $H \otimes H^*$ of a Hopf algebra and its dual (as in the case of the Drinfeld quantum double, or Schur functors with both multiplication endomorphisms and Foulkes derivatives, or the case of rational characters discussed in section 7), we note that the branching process does not provide an isomorphism of this extended structure, and hence the map from one to the other is a nontrivial transformation.

We now identify convenient bases of the dual Hopf algebras, Char-O* and Char-Sp* of the orthogonal and symplectic character Hopf algebras, Char-O and Char-Sp, respectively, and give explicit formulae for their structure maps. Since once more the orthogonal and symplectic cases work out similarly, we give only the orthogonal versions. Symplectic versions can be easily obtained by the usual recipe of changing the character brackets $[\ ] \mapsto \langle \ \rangle$ and interchanging series $A \leftrightarrow C$ and $B \leftrightarrow D$. 

28
**Proposition 6.19:** Let \( \text{Char}^* \) denote the Hopf algebra dual to \( \text{Char}\). Then a basis of \( \text{Char}^* \) is provided by the universal characters \( \lambda^* = \{ \lambda \cdot D \} \) which are such that

\[
[\lambda]^*([\mu]) := \langle [\lambda]^* | [\mu] \rangle = \delta_{\lambda,\mu}.
\]  

(92)

**Proof:**

\[
\langle [\lambda]^* | [\mu] \rangle = \langle \{ \lambda \} D | \{ \mu/C \} \rangle = \langle \{ \lambda \} \cdot DC | \{ \mu \} \rangle = \langle \{ \lambda \} | \{ \mu \} \rangle = \delta_{\lambda,\mu}.
\]

(93)

**Proposition 6.20:** The dual Hopf algebra \( \text{Char}^* \) is subject to the following structure maps:

- **product** \( m([\mu]^* \otimes [\nu]^*) = [\mu]^* \cdot [\nu]^* = [\mu \cdot \nu \cdot D]^* \)
- **unit** \( \eta(1) = [C]^* \) with \( [C]^* \cdot [\lambda]^* = [\lambda]^* = [\lambda]^* \cdot [C]^* \)
- **coproduct** \( \delta([\lambda]^*) = \sum_{\sigma,\zeta}([\lambda/\sigma] \cdot [\zeta]^* \otimes [\sigma \cdot \zeta]^*) \)
- **counit** \( \epsilon([\lambda]^*) = \delta_{\lambda,0} \)
- **antipode** \( S([\lambda]^*) = (-1)^{|\lambda|}[\lambda'BC]^* \)  

(94)

**Proof:** For the **product** we compute

\[
m([\mu]^* \otimes [\nu]^*) = m(\{ \mu \cdot D \} \otimes \{ \nu \cdot D \}) = \{ \mu \cdot D \cdot \nu \cdot D \} = \{ \mu \cdot \nu \cdot D \} = [\mu \cdot \nu \cdot D]^*.
\]

(95)

For the **unit** we just note that

\[
[C]^* = \{ C \cdot D \} = \{ 0 \},
\]

(96)

so that

\[
[C]^* \cdot [\lambda]^* = \{ 0 \} \cdot [\lambda \cdot D] = [\lambda] = [\lambda]^* \quad \text{and} \quad [\lambda]^* \cdot [C]^* = [\lambda \cdot D] \cdot \{ 0 \} = [\lambda \cdot D] = [\lambda]^*.
\]

(97)

A little more work is required for the **coproduct**

\[
\Delta([\lambda]^*) = \Delta(\{ \lambda \cdot D \}) = \Delta(\{ \lambda \}) \cdot \Delta(D) = \sum_{\sigma}([\lambda/\sigma] \otimes \{ \sigma \}) \cdot (D \otimes D) \cdot (\sum_{\zeta} \{ \zeta \} \otimes \{ \zeta \})
\]

\[
= \sum_{\zeta,\sigma}([\lambda/\sigma] \cdot [\zeta] \cdot D) \otimes ([\sigma \cdot \zeta] \cdot D) = \sum_{\zeta,\sigma}([\lambda/\sigma] \cdot [\zeta]^* \otimes [\sigma \cdot \zeta]^*),
\]

(98)
where the coproduct of $D$ has been taken from Proposition (2.1).

The counit maps as follows

$$
\epsilon([\lambda]^*) = \epsilon(\{\lambda\} \cdot D) = \delta_{\lambda,0}.
$$

(99)

While the explicit form of the antipode action for the dual Hopf algebra is given by

$$
S([\lambda]^*) = S(\{\lambda \cdot D\}) = (-1)^{|\lambda|}\{\lambda' \cdot D'\} = (-1)^{|\lambda|}\{\lambda' \cdot B \cdot CD\} = (-1)^{|\lambda|}[\lambda' \cdot BC]^*,
$$

(100)

where use has been made of the fact that all partitions in the set $D = 2\mathcal{P}$ are of even weight, and that $D' = B$

Remark. A dramatic difference between the character ring Hopf algebras for orthogonal and symplectic groups and their dual Hopf algebras is that the product maps of the former are filtered and hence contain only finitely many terms, as in (43). The dual character ring Hopf algebras, however, have products based on infinite Schur function series and acquire thereby an infinite number of terms, as in (95). Indeed, the basis elements of these dual Hopf algebras, $[\lambda]^* = \{\lambda\} \cdot D$ and $\langle \lambda \rangle^* = \{\lambda\} \cdot B$, clearly belong not to the ring $\Lambda$ but to the extension of $\Lambda$ to include infinite series of Schur functions. In fact these basis elements are the universal characters of lowest weight infinite-dimensional holomorphic discrete series irreducible representations of the $N \to \infty$ limit of the non-compact groups $SO^*(N)$ and $Sp(N,\mathbb{R})$ [21].

We add a few (more or less obvious) statements about this structure without explicit proof.

**Corollary 6.21**: The dual Hopf algebra $\text{Char} \cdot O^*$ is connected, that is we have:

$$
\Delta(\eta(1)) = \Delta([C]^*) = \Delta(\{0\}) = \{0\} \otimes \{0\} = [C]^* \otimes [C]^* = \eta(1) \otimes \eta(1);
$$

$$
\epsilon(m([\lambda]^* \otimes [\nu]^*)) = \epsilon(\{\lambda \cdot D \cdot \mu \cdot D\}) = \delta_{\lambda,0} \delta_{\mu,0} = \epsilon([\lambda]^*) \epsilon([\nu]^*).
$$

(101)

However, note that neither the product nor the coproduct is graded. On the other hand the connectedness property allows us to conclude that the antipode still is an antialgebra homomorphism (though we are bicommutative here), that is

$$
S(m([\lambda]^* \otimes [\mu]^*)) = S(\{\lambda \cdot D \cdot \mu \cdot D\}) = (-1)^{|\lambda|+|\mu|}\{\lambda' \cdot B \cdot \mu' \cdot B\}
$$

$$
= (-1)^{|\mu|}[\mu' \cdot BCD] (-1)^{|\lambda|}[\lambda' \cdot BCD]
$$

$$
= (-1)^{|\mu|}[\mu' \cdot BC]^* (-1)^{|\lambda|}[\lambda' \cdot BC]^* = S([\mu]^*) S([\lambda]^*),
$$

(102)

where once again use has been made of the fact that all partitions in the set $D = 2\mathcal{P}$ are of even weight and that $D' = B$. 

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The fact that the antipode fulfills its defining relation is established by noting that

\[ m(1 \otimes S)\Delta([\lambda]^*) = m(1 \otimes S) \left( \sum_{\zeta, \sigma} [\lambda/\sigma] \cdot [\zeta]^* \otimes [\sigma \cdot \zeta]^* \right) \]

\[ = m \left( \sum_{\zeta, \sigma} [\lambda/\sigma] \cdot [\zeta]^* \otimes (-1)^{\sigma+|\zeta|} [\sigma' \cdot \zeta' \cdot BC]^* \right) \]

\[ = \sum_{\zeta, \sigma} (-1)^{\sigma+|\zeta|} [\lambda/\sigma] \cdot [\sigma \cdot \zeta' \cdot BC' \cdot D]^* = \delta_{\lambda,0} \sum_{\zeta} (-1)^{|\zeta|} [\zeta \cdot \zeta' \cdot B]^* \]

\[ = \delta_{\lambda,0} [AC \cdot B]^* = \delta_{\lambda,0} [C]^* = \eta^*([\lambda]^*)\eta^*(1), \]  \hspace{1cm} (103)

as required. Use has been made of the antipode identity (40), CD = 1, AB = 1 and the fact that \( \sum_{\zeta} (-1)^{|\zeta|} \{\zeta \cdot \zeta'\} = AC \). This last identity can be established by comparing the dual Cauchy identity (29) with the product of the generating functions for the Schur function series A and C as given in (21).

### 7 Universal rational characters of the general linear group

There remain further finite-dimensional irreducible representations of these classical groups. For instance, in the case of GL(\(N\)), as well as the irreducible covariant tensor representations of highest weight \(\lambda\) having character

\[ \text{ch} V^{\lambda}_{\text{GL}(N)} = \{\lambda\}(x_1, \ldots, x_N) = s_{\mu}(x_1, \ldots, x_N), \]  \hspace{1cm} (104)

there exist irreducible contravariant tensor representations with highest weight \(\overline{\mu} = (\ldots, -\mu_2, -\mu_1)\) where \(\mu\) is a partition. These have character

\[ \text{ch} V^{\overline{\mu}}_{\text{GL}(N)} = \{\overline{\mu}\}(x_1, \ldots, x_N) = s_{\mu}(x_1, \ldots, x_N), \]  \hspace{1cm} (105)

with \(x_i = x_i^{-1}\) for \(i = 1, 2, \ldots, N\). More generally, there exist irreducible mixed tensor representations of GL(\(N\)) with highest weight \(\{\lambda; \overline{\mu}\} = (\lambda_1, \lambda_2, \ldots, 0, \ldots, 0, \ldots, -\mu_2, -\mu_1)\) where \(\lambda\) and \(\mu\) are both partitions. These representations have rational character [19, 22]

\[ \text{ch} V^{\lambda; \overline{\mu}}_{\text{GL}(N)} = \{\lambda; \overline{\mu}\}(x_1, \ldots, x_N; x_1, \ldots, x_N) = \sum_{\zeta \in \mathcal{P}} (-1)^{|\zeta|} s_{\lambda/\zeta}(x_1, \ldots, x_N)s_{\mu/\zeta'}(x_1, \ldots, x_N). \]  \hspace{1cm} (106)

It is straightforward to realise these characters as finite versions of certain universal rational characters defined in the ring \(\Lambda \otimes \overline{\Lambda}\) of symmetric functions with respect to. Their definition takes the form [22]

\[ \{\lambda; \overline{\mu}\} = \{\lambda\} \otimes \{\overline{\mu}\}/J = \sum_{\zeta \in \mathcal{P}} (-1)^{|\zeta|} \{\lambda/\zeta\} \otimes \{\mu/\zeta'\}. \]  \hspace{1cm} (107)
This has as its inverse the identity
\[
\{\lambda\} \otimes \{\overline{\mu}\} = \{\lambda; \overline{\mu}\}/K = \sum_{\eta \in P} \{\lambda/\eta; \mu/\eta\}.
\] (108)

To be more explicit, in terms of two denumerably infinite sequences of indeterminates, say \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\) we have
\[
\{\lambda; \overline{\mu}\}(x; y) = \sum_{\zeta \in P} (-1)^{|\zeta|} s_{\lambda/\zeta}(x) s_{\mu/\zeta'}(y),
\] (109)
from which we recover our mixed tensor irreducible characters in the form
\[
\text{ch } V^{\lambda, \overline{\mu}}_{\text{GL}(N)} = \{\lambda; \overline{\mu}\}(x_1, \ldots, x_N, 0, \ldots, 0; x_1, \ldots, x_N, 0, \ldots, 0) = \sum_{\zeta \in P} (-1)^{|\zeta|} s_{\lambda/\zeta}(x_1, \ldots, x_N) s_{\mu/\zeta'}(\overline{x}_1, \ldots, \overline{x}_N),
\] (110)
where we have exploited the usual stability properties of Schur functions with respect to vanishing indeterminates.

The notation in (107) and (108) is such that in \(\Lambda \otimes \overline{\Lambda}\) we have:
\[
J = J_1(x, \overline{y}) = \prod_{i,j} (1 - x_i \overline{y}_j) = \sum_{\zeta} (-1)^{|\zeta|} s_{\zeta}(x) s_{\zeta'}(\overline{y}) = \sum_{\zeta} (-1)^{|\zeta|} \{\zeta\} \otimes \{\overline{\zeta}\} ;
\] (111)
\[
K = K_1(x, \overline{y}) = \prod_{i,j} (1 - x_i \overline{y}_j)^{-1} = \sum_{\zeta} s_{\zeta}(x) s_{\zeta'}(\overline{y}) = \sum_{\zeta} \{\zeta\} \otimes \{\overline{\zeta}\} .
\] (112)
where use has been made of the Cauchy identity and its dual. Their coproducts take the form \(\Delta(J) = (J \otimes J) \cdot J'\) and \(\Delta(K) = (K \otimes K) \cdot K'\) with their cut coproducts given by
\[
J' = \sum_{\sigma, \tau \in P} (-1)^{|\sigma|+|\tau|} \left(\{\sigma\} \otimes \{\overline{\tau}\}\right) \otimes \left(\{\tau'\} \otimes \{\overline{\sigma'}\}\right) ;
\] (113)
\[
K' = \sum_{\sigma, \tau \in P} \left(\{\sigma\} \otimes \{\overline{\tau}\}\right) \otimes \left(\{\tau\} \otimes \{\overline{\sigma}\}\right).\] (114)
This can be seen by taking the product forms of \(J\) and \(K\) and mapping \(x\) to \((x, u)\) and \(\overline{y}\) to \((\overline{y}, \overline{v})\). Separating off the products over \(x_i \overline{y}_j\) and \(u_k \overline{v}_\ell\) leaves products over \(x_i \overline{v}_\ell\) and \(\overline{y}_j u_k\) that can be expanded once again using the Cauchy identity (28) and its dual (29) to give the required result.

The universal rational characters \(\{\lambda; \overline{\mu}\}\) for all partitions \(\lambda\) and \(\mu\) form a basis of \(\Lambda \otimes \overline{\Lambda}\). Moreover we have
**Theorem 7.22:** The algebra $\text{Char-GLrat}$ generated by the universal rational characters $\{\lambda; \overline{\nu}\}$ is a bicommutative Hopf algebra. Its structure maps are given by

- **product** $m(\{\kappa; \overline{\lambda}\}, \{\mu; \overline{\nu}\}) = \{\kappa; \overline{\lambda}\} \cdot \{\mu; \overline{\nu}\} = \sum_{\sigma, \tau \in \mathcal{P}} \{(\kappa/\sigma) \cdot (\mu/\tau) ; (\overline{\lambda}/\tau) \cdot (\overline{\nu}/\sigma)\}$

- **unit** $\eta(1) = \{(0); (0)\}$

- **coproduct** $\Delta(\{\mu; \overline{\nu}\}) = \sum_{\sigma, \tau, \rho \in \mathcal{P}} \{\mu/\sigma; \overline{\nu}/\tau\} \otimes \{\sigma/\rho; \overline{\tau}/\rho\}$

- **counit** $\epsilon(\{\mu; \overline{\nu}\}) = \delta_{\mu,(0)} \delta_{\nu,(0)}$

- **antipode** $S(\{\mu; \overline{\nu}\}) = (-1)^{|\mu|+|\nu|} \sum_{\rho} \{\mu'/\rho; \overline{\nu'/\rho}\}$

- **scalar product** $\langle \cdot | \cdot \rangle_{1,T}(\{\kappa; \overline{\lambda}\} \otimes \{\mu; \overline{\nu}\}) = \langle \{\kappa; \overline{\lambda}\} | \{\mu; \overline{\nu}\} \rangle_{1,T} = \delta_{\kappa,\mu} \delta_{\lambda,\nu}$. (115)

The product formula of Theorem 7.22 was originally given as the rule for decomposing products of irreducible mixed tensors of $\text{GL}$ by Abramsky and King [1, 17, 19]. Its derivation can be accomplished most easily within the framework of the current paper by noting, precisely as in the derivation of the Newell-Littlewood product formula (43), that $\{\kappa; \overline{\lambda}\} \cdot \{\mu; \overline{\nu}\} = (((\{\kappa\} \otimes \{\overline{\lambda}\})/J) \cdot ((\{\mu\} \otimes \{\overline{\nu}\})/J) = (((\{\kappa\} \otimes \{\overline{\lambda}\})/J) \cdot ((\{\mu\} \otimes \{\overline{\nu}\})/J)/K$ where the coefficient of $\{\eta; \overline{\zeta}\}$ in this last expression is given by

$$\langle \{\eta\} \otimes \{\overline{\zeta}\} | (((\{\kappa\} \otimes \{\overline{\lambda}\})/J) \cdot ((\{\mu\} \otimes \{\overline{\nu}\})/J))/K \rangle$$

$$= \langle \{\eta\} \otimes \{\overline{\zeta}\} \cdot K | (((\{\kappa\} \otimes \{\overline{\lambda}\})/J) \cdot ((\{\mu\} \otimes \{\overline{\nu}\})/J)) \rangle$$

$$= \langle \Delta(\{\eta\} \otimes \{\overline{\zeta}\}) \cdot (K \otimes K') | (((\kappa\} \otimes \{\overline{\lambda}\})/J) \otimes ((\{\mu\} \otimes \{\overline{\nu}\})/J)) \rangle$$

$$= \langle \Delta(\{\eta\} \otimes \{\overline{\zeta}\}) \cdot K' | (((\kappa\} \otimes \{\overline{\lambda}\})/JK) \otimes ((\{\mu\} \otimes \{\overline{\nu}\})/(JK)) \rangle$$

$$= \langle \Delta(\{\eta\} \otimes \{\overline{\zeta}\}) \cdot K' | (((\kappa\} \otimes \{\overline{\lambda}\})/(JK) \otimes ((\{\mu\} \otimes \{\overline{\nu}\})/(JK)) \rangle$$

$$= \sum_{\sigma, \tau \in \mathcal{P}} \langle \Delta(\{\eta\} \otimes \{\overline{\zeta}\}) | (((\kappa/\sigma) \otimes \{\overline{\lambda}/\tau\}) \otimes ((\{\mu/\tau\} \otimes \{\overline{\nu}/\sigma\})) \rangle$$

$$= \sum_{\sigma, \tau \in \mathcal{P}} \langle \{\eta\} \otimes \{\overline{\zeta}\} | (((\kappa/\sigma) \cdot (\mu/\tau) \otimes ((\overline{\lambda}/\tau) \cdot \overline{\nu}/\sigma)) \rangle,$$ (116)

which gives the product rule of Theorem 7.22.
The corresponding result for the evaluation of coproducts coincides with the large $M$ and $N$ limit of the branching rule formula for the restriction from $\text{GL}(M+N)$ to $\text{GL}(M) \times \text{GL}(N)$ [18] and may be derived as follows:

$$
\Delta(\{\mu;\nu\}) = \sum_{\zeta \in \mathcal{P}} (-1)^{\mid \zeta \mid} \Delta(\{\mu/\zeta\}) \otimes \Delta(\{\nu/\zeta\})
$$

$$
= \sum_{\zeta,\sigma,\tau \in \mathcal{P}} (-1)^{\mid \zeta \mid} (\{\mu/\zeta \sigma\} \otimes \{\sigma\}) \otimes (\{\nu/\zeta \tau\} \otimes \{\tau\})
$$

$$
= \sum_{\zeta,\sigma,\tau \in \mathcal{P}} (-1)^{\mid \zeta \mid} (\{\mu/\sigma \zeta\} \otimes \{\sigma/\tau \zeta\}) \otimes (\{\sigma\} \otimes \{\tau\})
$$

$$
= \sum_{\sigma,\tau \in \mathcal{P}} \{\mu/\sigma;\nu/\tau\} \otimes \sum_{\rho \in \mathcal{P}} \{\sigma/\rho;\tau/\rho\},
$$

which coincides with the required result for the coproduct given in Theorem 7.22.

With these two results it is straightforward but rather tedious to verify that the unit, counit and antipode structure maps displayed in Theorem 7.22 satisfy all the requirements of a Hopf algebra including the bialgebra and antipode conditions.

### 8 Conclusions and discussion

Our treatment shows that on the Hopf algebraic side the two character ring Hopf algebras $\text{Char-O}$ and $\text{Char-Sp}$ behave in exactly the same way. They share the same product structure and differ only in the coproduct where the series $D$ and $B$ are involved. This stems from the fact that the deformation of the product actually depends only on the proper cut part $\Delta'$ of the coproduct

$$
\Delta'(a) = \Delta(a) - 1 \otimes a - a \otimes 1.
$$

These proper cut parts of $\Delta'({\{2\}})$ and $\Delta'({\{11\}})$ are identical (simply the single term $\{1\} \otimes \{1\}$), producing the same deformation. As shown in [13] this is no longer true for deformations based on tensors of higher degree. For example $\Delta'({\{3\}})$, $\Delta'({\{21\}})$ and $\Delta'({\{111\}})$ are all different.

The orthogonal and symplectic character of the underlying group finds its counterpart in the proper definition of the various symmetric function bases. While the primitives look similar, complete and orthogonal symmetric functions differ. This is important for applications in physics, since orthogonal, elementary and power sum symmetric functions can be used to encode partition functions of physical systems [33, 34]. Assuming one has a gas of particles, say atoms or even molecules, having an internal orthogonal or symplectic symmetry, one is naturally led to the bases defined in the previous sections.

We have been able with this approach to evaluate products and coproducts within the Hopf algebras of the universal character rings of the orthogonal and symplectic groups, thereby constructing explicit formulae for the decomposition of products of representations of these groups and of the restriction of these representations to a variety of subgroups.
To recover the irreducible characters of $O(N)$ and $Sp(N)$ in the finite $N$ case one merely limits the arguments of the universal characters to the eigenvalues of the relevant group elements $g$ supplemented by zeros. Denoting the eigenvalues by $x_k$ and $\overline{x}_k = x_k^{-1}$ for $k = 1, 2, \ldots, K$, together with $\pm 1$ and $x_{2K+1}$, as appropriate, one obtains:

\[
\begin{align*}
\text{ch} V^\lambda_{O(2K)} &= [\lambda](x_1, \ldots, x_K, \overline{x}_1, \ldots, \overline{x}_K, 0, \ldots, 0) & \text{for } g \in SO(2K) ; \\
\text{ch} V^\lambda_{O(2K)} &= [\lambda](x_1, \ldots, x_{K-1}, \overline{x}_1, \ldots, \overline{x}_{K-1}, 1, -1, 0, \ldots, 0) & \text{for } g \notin SO(2K) ; \\
\text{ch} V^\lambda_{O(2K+1)} &= [\lambda](x_1, \ldots, x_K, \overline{x}_1, \ldots, \overline{x}_K, 1, 0, \ldots, 0) & \text{for } g \in SO(2K+1) ; \\
\text{ch} V^\lambda_{O(2K+1)} &= [\lambda](x_1, \ldots, x_K, \overline{x}_1, \ldots, \overline{x}_K, -1, 0, \ldots, 0) & \text{for } g \notin SO(2K+1) ; \\
\text{ch} V^\lambda_{Sp(2K)} &= (\lambda)(x_1, \ldots, x_K, \overline{x}_1, \ldots, \overline{x}_K, 0, \ldots, 0) & \text{for } g \in Sp(2K) ; \\
\text{ch} V^\lambda_{Sp(2K+1)} &= (\lambda)(x_1, \ldots, x_K, \overline{x}_1, \ldots, \overline{x}_K, x_{2K+1}, 0, \ldots, 0) & \text{for } g \in Sp(2K+1) ,
\end{align*}
\]

where the final character of $Sp(2K + 1)$ is indecomposable, rather than irreducible, with the first $2K$ eigenvalues being those of an element of $Sp(2K)$ and the $x_{2K+1}$ being an element of $GL(1)$.

In order to exploit to the full the results on products and branchings implied by the properties of the universal character rings $\text{Char}-O$ and $\text{Char}-Sp$ in the context of the groups $O(N)$ and $Sp(N)$ it is necessary to invoke certain modification rules [28, 17, 4, 23] that apply to the above characters whenever the length $\ell(\lambda)$ of the partitions $\lambda$ exceeds $K$, or in the case of $SO(2K)$ is equal to $K$.

This is also necessary in dealing with the restriction of the universal rational characters to the case of $GL(N)$ for finite $N$. As already noted, if we denote the eigenvalues of the group element of $GL(N)$ by $x_k$ for $k = 1, 2, \ldots, N$, then the corresponding characters are given by

\[
\text{ch} V^\lambda_{GL(N)} = \{\lambda; \overline{\mu}\}(x_1, \ldots, x_N, 0, \ldots, 0; x_1, \ldots, x_N, 0, \ldots, 0) .
\]

The corresponding modification rules have been described elsewhere [17, 4, 22].

There remain finite dimensional irreducible spin and indeed spinor representations of the orthogonal groups that we have not discussed here. It is possible to make earlier developments [18, 4] more complete and rigorous by defining, following Okada [30], not only spin characters but also universal spinor characters in the original ring, $\Lambda$, but now with coefficients in $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$. However, the double-valued spin characters do not lend themselves in general to a Hopf algebra analysis because of the complications that arise from the necessity of distinguishing not only the even and odd $O(2K)$ and $O(2K + 1)$ cases, but also the effect of products, coproducts and the antipode map on the spin representations of group elements of determinant $+1$ and $-1$. In addition, consideration of the characters of such spin representations of dimension $2^K$ would take us outside our chosen domain of symmetric functions.

Applications of symmetric function techniques are widespread. We have argued in this paper that it is important to pursue the Hopf algebraic machinery behind the character Hopf algebras, to generalize these techniques to form a powerful tool which can deal with more general sub-symmetries than orthogonal and symplectic ones. We restricted our studies here to the classical cases, but even there novel points arose.
We believe that the general branching scenario is quite universal, and have proposed to take it as a blueprint for quantum field calculations [11, 12]. The present work is preparatory to the study of the character ring Hopf algebras at a (conformal) quantum field level using vertex operator techniques. It has already allowed the construction of vertex operators for both the orthogonal and symplectic groups as well as an extension of such constructions to more general subgroups of the general linear group GL [14].

Finally we reiterate that the presently developed machinery was obtained by literally redoing the quantum field theory calculations done in [7, 5], in the context of symmetric functions. We hope to show elsewhere, that the insights gained here can in turn be profitably applied in quantum field theory, clarifying algebraic constructions from a group representation point of view. In particular non-classical subgroup branchings may lead to new methods allowing the computation of nontrivial, that is non-quadratic, invariants, and offering the possibility of new frameworks for general interacting quantum field theories.

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